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NONLINEAR DELAY-DIFFERENTIAL EQUATIONS WITH SMALL LAG

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ABSTRACT. Asymptotic formulae for the solutions of nonlinear functional differential system are obtained.

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1 Introduction

Let $q \geq 0$ be a constant and let $C_0 = C([-q,0],\mathbf{R}^n)$ be the Banach space of continuous functions $\varphi : [-q,0] \to \mathbf{R}^n$ equipped with the norm

$$\|\varphi\| = sup_{-q\leq s\leq 0}|\varphi(s)|.$$

For $y \in C([t-q,t],\mathbf{R}^n)$, we denote by y_t the element of C_0 defined by

$$y_t(s) = y(t+s), \quad -q \leq s \leq 0.$$

We will also denote, for $y \in C([t-2q,t],\mathbf{R}^n), y^t$ the functional defined by

$$y^t(s) = y(t+s), \quad -2q \leq s \leq 0$$

for which we consider the norm:

$$\|\varphi\|_2 = sup_{-2q\leq s\leq 0}|\varphi(s)|.$$

Consider $F : [0,\infty) \times C_0 \to \mathbf{R}^n$ and $g : [0,\infty) \times \mathbf{R}^n \to \mathbf{R}^n$ two continuous functions satisfying the "closeness" condition

(C) There exists a continuous function $\lambda : [0,\infty) \to [0,\infty)$ such that

$$|F(t,\varphi) - g(t,\varphi(0))| \leq \lambda(t)\|\varphi'\| \tag{1.1}$$

for any continuously differentiable function $\varphi : [-q,0] \to \mathbf{R}^n$.

We Remark that (1.1) holds with $\|\varphi'\|$ and not with $\|\varphi\|$. See [15].

We wish to study the relation between the solutions of the functional differential system

$$y'(t) = F(t, y_t) \tag{1.2}$$

and the solutions of the ordinary differential system

$$x'(t) = g(t, x(t)) \tag{1.3}$$

For system (1.3) we suppose that the following condition is fulfilled:

(G) The derivative of $g : g_x = g_x(t, x)$ exists and is continuous on $[0, \infty) \times \mathbf{R}^n$. System (1.3) is an h-system in variation with radius of attraction δ, where $h : [0, \infty) \to (0, \infty)$ is a continuous function.

We recall that a system (1.3) or its null-solution is an h-system in variation [5, 6] with radius of attraction δ, if there exist a continuous function $h : [0, \infty) \to (0, \infty)$ and constants $K \geq 1$ and $\delta > 0$ such that for $0 \leq |x_0| < \delta$ we have

$$|\Phi(t, t_0, x_0)| \leq K h(t) h(t_0)^{-1} \qquad (t \geq t_0 \geq 0),$$

where $\Phi(t, t_0, x_0)$ is the fundamental matrix of the variational system

$$z'(t) = g_x(t, x(t, t_0, x_0)) z(t)$$

such that $\Phi(t, t_0, x_0) = Id$ (the identity matrix). Here $x = x(t, t_0, x_0)$ represents the solution x passing throught the point (t_0, x_0).

This problem appears in Bellman [1] who proposed to investigate conditions on the lag r to know the behavior of solutions of the functional differential equation

$$u'(t) + a u(t - r(t)) = 0, \qquad a \text{ constant} \tag{1.4}$$

when $r(t) \to 0$ as $t \to \infty$. In [2], Cooke proves that if $a > 0$ and $r \in L_1([0, \infty))$ then any solution u of (1.1) satisfies

$$u(t) = e^{-at}[c + o(1)], \qquad t \to \infty$$

for some constant c. In [3], Cooke generalizes this result to linear systems of functional differential equations asymptotically autonomous. Grossman and Yorke [4] consider the one-dimensional functional differential equation

$$u'(t) = a(t) u(t - r(t)).$$

In [10] we have extended some of these results to the scalar functional equation

$$u'(t) = a(t) u(t - r(t, u))$$

with a lag of implicit type, generalizing the case

$$u'(t) = -au(t - r(u(t)))$$

studied by Cooke [4]. See also [12, 14]. We note that in all of these cases the estimate (1.1) does not hold with $\|\varphi\|$ instead of $\|\varphi'\|$.

In this paper, for the nonlinear problem (1.2), we obtain the relation

$$y = x + h \cdot \tilde{o}(1),$$

between the solutions y of (1.2) and x of (1.3), where $\tilde{o}(1)$ is a convergent function as $t \to \infty$. We will prove also that the nonlinear functional system (1.2) is an h-system (see Remark 1).

As an application we get asymptotic formulae of the solutions of second order delay equation [11, 13]

$$y'' + c(t)y(t - r(t)) = 0 \tag{1.5}$$

in terms of the solutions of

$$z'' + c(t)z = 0, \tag{1.6}$$

extending ordinary results [7, 8].

2 Main Results

In this section we get asymptotic formulae for the solutions of system (1.2). We denote by $y = y(t; t_0, y_{t_0})$ a solution y of Eq. (1.2) with initial function $y_{t_0} \in C_0$.

Theorem 1 *In addition to conditions (C) and (G),assume:*

(i) There exists a continuous and nonnegative function $c(t)$ such that

$$|F(t, \varphi)| \leq c(t)\|\varphi\|$$

for all $t \geq 0$ and all $\varphi \in C_0$.

(ii) $\beta(t)\lambda(t)\|c_t\| \in L_1([0, \infty))$, where $\beta(t) = h(t)^{-1}\|h^t\|_2$.

Then for any solution $y = y(t; t_o, y_{t_0})$ of (1.2) with $\|y_{t_0}\| \leq \delta$ there exists a solution x of (1.3) such that

$$y = x + h \cdot \tilde{o}(1),$$

where $\tilde{o}(1)$ is a function defined on $[t_0, \infty)$ which converges as $t \to \infty$.

Proof. By condition (G), for $|y(t_0)| \leq \delta$, the solution $x = x(t; t_0, y(t_0))$ of the ordinary system (1.3) is well defined and satisfies $|x(t; t_0, y(t_0))| \leq K|y(t_0)|h(t)h(t_0)^{-1}$ for $t \geq t_0 \geq 0$ and $K \geq 1$ a constant. Now, by (i), the solution $y = y(t, t_0, y_{t_0})$ of system (1.2) is defined on $[t_0 - q, \infty)$. By the formula of variation of the constants, we have for $t \geq t_1 \geq t_0$

$$y(t) = x(t; t_1, y(t_1)) + \int_{t_1}^{t} \Phi(t, s, y(s))[F(s, y_s) - g(s, y(s))]ds \tag{2.1}$$

Then, by (C) and (G)

$$|y(t)| \leq K|y(t_1)|h(t)h(t_1)^{-1} + Kh(t)\int_{t_1}^{t} h(s)^{-1}\lambda(s)\|y_s'\|ds$$

or

$$h(t)^{-1}|y(t)| \leq Kh(t_1)^{-1}|y(t_1)| + K\int_{t_1}^{t} \lambda(s)h(s)^{-1}\|y_s'\|ds.$$

Thus $z(t) = h(t)^{-1}|y(t)|$ satisfies

$$z(t) \leq Kz(t_1) + \int_{t_1}^{t} K\lambda(s)h(s)^{-1}\|y_s'\|ds \tag{2.2}$$

For $u \in [-q, 0]$ and $s \geq t_1$, by (i), we have

$$|y_s'(u)| = |F(s+u, y_{s+u})| \leq c_s(u)\|y_{s+u}\| = c_s(u)|y(v)|$$

for some $v = v(s) \in [s - 2q, s]$. Further

$$h(s)^{-1}|y(v)| = h(s)^{-1}h(v)z(v) \leq \beta(s)z(v).$$

Thus

$$h(s)^{-1}\|y_s'\| \leq \beta(s)\|c_s\|m(s), \tag{2.3}$$

where $m(t) = \max_{t-2q \leq s \leq t} |z(s)|$. Substituting this into (2.2) we obtain

$$z(t) \leq Kz(t_1) + \int_{t_1}^{t} K\lambda(s)\beta(s)\|c_s\|m(s)ds. \tag{2.4}$$

Since the right member of (2.4) is increasing as a function in t, for $t \geq t_1 + 2q$ we have $m(t) \leq Kz(t_1) + \int_{t_1}^{t} Kr(s)\beta(s)\|c_s\|m(s)ds$. Then by (ii), Gronwall's inequality implies that m and hence z are bounded. Moreover, for any t fixed $\Phi(t, s, y(s))[F(s, y_s) - g(s, y(s))] \in L_1([0, \infty))$ as a function of s because from (C), (G), (ii) and (2.3) we get

$$\begin{aligned} |\Phi(t, s, y(s))[F(s, y_s) - g(s, y(s))]| &\leq Kh(t)h(s)^{-1}\lambda(s)\|y_s'\| \\ \leq K_1 h(t)\|c_s\|\lambda(s)\beta(s)m(s) &\leq K_2 h(t)\lambda(s)\beta(s)\|c_s\| \in L_1([0, \infty)). \end{aligned}$$

Then the integral in (2.1) can be written as $h(t) \cdot \tilde{o}(1)$, where $\tilde{o}(1)$ denotes a function of t which has a limit as $t \to \infty$. The proof is complete.

Remark 1. Since we have proved $h(t)^{-1}|y(t)| \leq m(t) \leq KK_1|z(t_1)| = KK_1h(t_1)^{-1}|y(t_1)|$ for $t \geq t_1 \geq t_0$ and K_1 a positive constant, we have also established

$$|y(t)| \leq K_2h(t)h(t_1)^{-1}|y(t_1)|, (t \geq t_1 \geq t_0), K_2 \text{ constant}$$

that is, the nonlinear functional system (1.2) is also an h-system.

Theorem 1 includes the interesting type of equations as:

$$y' = F(t, y(t) - y(t - r(t))), \tag{2.5}$$

where $r : [0, \infty) \to [0, q]$ is a continuous function.

For this equation, system (1.3) becomes $x' = 0$ and (1.1) becomes

$$|F(t, \varphi)| \leq r(t)\|\varphi'\| \tag{2.6}$$

Thus here $h \equiv 1$, $\beta \equiv 1$ and we have

Corollary 1 *Assume that (i) of Theorem 1 and (2.6) hold. If* $r(t) \cdot \|c_t\| \in L_1([0, \infty))$, *then any solution* $y = y(t; t_0, y_{t_0})$ *of (2.5) there exists a constant vector* v *such that*

$$y = y(t_0) + v + o(1)$$

as $t \to \infty$. *In particular, any solution of (2.5) is asymptotically constant.*

Proceeding as in the proof of the Theorem 1, with a Bihari's inequality, Corollary 1 can be obtained for the nonlinear equations

$$y' = y^3(t) - y^3(t - r(t)) \quad \text{or} \quad y' = [y(t) - y(t - r(t))]^3$$

since in this case we have an estimate of the type:

$$|F(t, g)| \leq Kr(t)w(\|g'\|), \tag{2.7}$$

where $w : (0, \infty) \to (0, \infty)$ is a continuous, nondecreasing function satisfying $w(0) \geq 0$ and

$$\int_{0+}^{1} \frac{ds}{w(s)} = \infty \tag{2.8}$$

Thus from lemma 1, [6] we obtain:

Corollary 2 *Under the conditions of Corollary 1 with (2.7-2.8) instead of (2.6), there exists a constant* $\rho > 0$ *such that any solution* $y = y(t; t_0, y_{t_0})$ *with* $\|y_{t_0}\| \leq \rho$ *is defined on* $[t_0 - q, \infty)$ *and*

$$y = y(t_0) + v(t_0) + o(1), \qquad t \to \infty \tag{2.9}$$

where $v = v(t_0)$ *is a constant vector such that* $v(t_0) \to 0$ *as* $t_0 \to \infty$. *Moreover,* $\rho = \rho(t_0)$ *verifies* $\rho(t_0) \to \infty$ *as* $t_0 \to \infty$. *Then if* t_0 *is chosen large enough for any initial function* φ *there exists* t_0 *large enough such that the solution* $y = y(t, t_0, \varphi)$ *verifies the above asymptotic formulae.*

Some simple consequences are the following:

Corollary 3 *If, for* $h(t) = exp(\int_0^t a(s)ds)$, $a\|a_t\|h(t)^{-1}\|h^t\|_2 r \in L_1([0,\infty))$, *then the solutions of the scalar equation*

$$y'(t) = a(t)y(t - r(t)),$$

satisfy

$$y(t) = exp(\int_0^t a(s)ds)[c + o(1)], \qquad c \text{ constant.}$$

Thus, in particular, the solutions of

$$y'(t) = -ty(t - e^{-3t})$$

and

$$y'(t) = ty(t - r(t)), t^2 r(t) \in L_1([0,\infty)),$$

satisfy respectively,

$$y = e^{-t^2/2}[c + o(1)], \qquad c \text{ constant}$$

and

$$y = e^t[c + o(1)], \qquad c \text{ constant.}$$

Now, an explicite nonlinear scalar example is shown. Let $g(t,x) = -e^t x^3$ in equation (1.3):

$$x'(t) = -e^t x^3(t)$$

This ordinary system has the solutions

$$x(t, t_0, x_0) = \frac{|x_0|}{(1 + 2x_0^2(e^t - e^{t_0}))^{1/2}}$$

whence it is an h-system with $h(t) = e^{-t/2}$. Then, Theorem 1 implies that the solutions $y = y(t, t_0, y_{t_0})$ of the scalar equation

$$y'(t) = -e^t y^3(t - e^{-\alpha t}), \alpha > 2,$$

satisfy

$$y(t) = x(t) + e^{-t/2} \cdot \tilde{o}(1),$$

for t large enough.

Corollary 4 *If A is a stable matrix, then any solution of*

$$y' = Ay(t - r(t)), \quad r \in L_1([0, \infty))$$

satisfies

$$y = e^{tA}x_0 + e^{-\alpha t} \cdot \tilde{o}(1)$$

where x_0 is a constant vector, $0 > \alpha > maxRe\lambda$ for λ an eigenvalue of A and $\tilde{o}(1)$ is a convergent vector as $t \to \infty$.

When (1.3) is a linear and an h-system (see [6]) we have:

Corollary 5 *If system*

$$x' = A(t)x \tag{2.10}$$

is an h-system and $rh(t)^{-1}\|h^t\|_2\|A\|\|A_t\| \in L_1([0, \infty))$, then for any solution y of

$$y' = A(t)y(t - r(t))$$

satisfies

$$y = \Phi y_0 + h\tilde{o}(1) \quad as \quad t \to \infty$$

where y_0 is a constant vector and Φ is a fundamental matrix of (2.10).

3 An application: Asymptotic formulae for the solutions of (1.5)

Consider the functional differential equation

$$y'' + c(t)y(t - r(t)) = 0 \tag{3.1}$$

where $c : [0, \infty) \to \mathbf{R}$ and $r : [0, \infty) \to [0, q]$ are continuous functions.

As usually, a solution of eq. (3.1) is a function $y = y(t; t_0, \varphi, \psi)$ such that y satisfies the delay-differential equations (3.1) and

$$y_{t_0} = \varphi, \quad y'_{t_0} = \psi,$$

where $\varphi, \psi \in C([-q, 0], \mathbf{R})$.

For $r = r(t)$ small, in some sense which will be precised, we hope that the solutions y of Eq (3.1) behave asymptotically as the solutions z of the ordinary differential equation

$$z'' + c(t)z(t) = 0. \tag{3.2}$$

We will prove that any solution y of Eq (3.1) are defined on all of $I = [0, \infty)$ and it satisfies as $t \to \infty$:

$$\begin{aligned} y &= (\delta_1 + o(1))z_1 + (\delta_2 + o(1))z_2 \\ y' &= (\delta_1 + o(1))z_1' + (\delta_2 + o(1))z_2' \end{aligned} \tag{3.3}$$

where $\{z_1, z_2\}$ is a fundamental system of solutions of Eq (3.2) and $\{\delta_1, \delta_2\}$ are constants. Let

$$y(t) = A(t)z_1(t) + B(t)z_2(t) \tag{3.4}$$

under the condition

$$A'z_1 + B'z_2 = 0 \tag{3.5}$$

Then, we have $y' = Az_1' + Bz_2'$ and $y'' = A'z_1' + B'z_2' + Az_1'' + Bz_2''$. Thus $y'' = A'z_1' + B'z_2' - c(Az_1 + Bz_2)$. Therefore

$$A'z_1' + B'z_2' = c(t)[y(t) - y(t - r(t))]. \tag{3.6}$$

Solving Eqs. (3.5) and (3.6), we get

$$\begin{aligned} A' &= -w^{-1}z_2 \cdot c(t)[y(t) - y(t - r(t))] \\ B' &= w^{-1}z_1 \cdot c(t)[y(t) - y(t - r(t))] \end{aligned} \tag{3.7}$$

where w is the Wronskian of system $\{z_1, z_2\}$. Now, we have

$$\begin{aligned} |y(t) - y(t - r(t))| &= |\textstyle\int_{t-r(t))}^{t} y'(s)ds| = |\int_{-r(t)}^{0} y'(t + s)ds| \\ &= |\textstyle\int_{-r(t)}^{0} y_t'(s)ds| = |\int_{-r(t)}^{0} (Az_1' + Bz_2')_t(s)ds|. \end{aligned}$$

Thus

$$|y(t) - y(t - r(t))| \leq r(t) \max_{i=1,2} \|z_{it}'\| \cdot (\|A_t\| + \|B_t\|).$$

Then, by system (3.7), the vector $x = (A, B)$ satisfies a system of functional differential equations of the type

$$x' = F(t, x_t) \tag{3.8}$$

satisfying the conditions (i) $F : I \times C_0 \to \mathbf{R}$ is a continuous function (ii) $|F(t, \varphi)| \leq \lambda(t)\|\varphi\|$, $(t, \varphi) \in I \times C_0$.

In this point, we need the following Theorem concerning the asymptotic behavior of system (3.8).

Theorem 2 *Assume the above conditions (i) and (ii), where* $\lambda \in C(I, \mathbf{R})$ *satisfies* $\lambda(t) \in L_1(I)$. *Then the solutions with continuous initial conditions of Eq (3.8) are defined on all of* I *and they converge as* $t \to \infty$.

The proof of this theorem is omitted because it is similar to that of Theorem 1.

Thus, we get:

Theorem 3 *Assume that* $r(t)|c(t)| \cdot |z_i(t)| \cdot \|z'_{it}\| \in L_1(I)$ $i = 1, 2$. *Then any solution* $y = y(t; t_0, y_{t_0}, y'_{t_0})$ *satisfies formulae (3.3)* .

Proof. The application of Theorem 2 implies that A and B converge as $t \to \infty$. The formulae (3.3) follow from (3.4) and $y' = Az'_1 + Bz'_2$.

So, we have

Corollary 6 *If* $r \in L_1(I)$, *then any solution* y *of the functional differential equation*

$$y'' + ay(t - r(t)) = 0, \qquad a > 0 \quad constant$$

satisfies for $t \to \infty$,

$$y = (\delta_1 + o(1))sinat + (\delta_2 + o(1))cosat$$

$$y' = a(\delta_1 + o(1))cosat - a(\delta_2 + o(1))sinat$$

More generally, using Green-Liouville formulae ([7]) for the solutions of (3.2) we get:

Corollary 7 *If* $c(t) \in C^2(I)$, $c > 0$ *and* $c^{-3/2}c''$, $r(t) \cdot |c^{3/4}(t)|\|c_t^{1/4}\| \in L_1(I)$ *then any solution* y *of the functional differential equation*

$$y'' + c(t)y(t - r(t)) = 0$$

satisfies for $t \to \infty$

$$y = c(t)^{-1/4}[(\delta_1 + o(1))exp(i\int^t c^{1/2}(s)ds) + (\delta_2 + o(1))exp(-i\int^t c^{1/2}(s)ds)]$$

$$y' = c(t)^{1/4}[i(\delta_1 + o(1))exp(i\int^t c^{1/2}(s)ds) + i(\delta_2 + o(1))exp(-i\int^t c^{1/2}(s)ds)]$$

For more related results, see [9].

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