NONLINEAR DELAY-DIFFERENTIAL EQUATIONS WITH SMALL LAG

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ABSTRACT. Asymptotic formulae for the solutions of nonlinear functional differential system are obtained.

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1 Introduction

Let $q > 0$ be a constant and let $C_0 = C([-q, 0], \mathbb{R}^n)$ be the Banach space of continuous functions $\varphi : [-q, 0] \rightarrow \mathbb{R}^n$ equipped with the norm

$$
\|\varphi\| = \sup_{-q \leq s \leq 0} |\varphi(s)|.
$$

For $y \in C([t - q, t], \mathbb{R}^n)$, we denote by $y_t$ the element of $C_0$ defined by

$$
y_t(s) = y(t + s), \quad -q \leq s \leq 0.
$$

We will also denote, for $y \in C([t - 2q, t], \mathbb{R}^n)$, $y'$ the functional defined by

$$
y'(s) = y(t + s), \quad -2q \leq s \leq 0
$$

for which we consider the norm:

$$
\|\varphi\|_2 = \sup_{-2q \leq s \leq 0} |\varphi(s)|.
$$

Consider $F : [0, \infty) \times C_0 \rightarrow \mathbb{R}^n$ and $g : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ two continuous functions satisfying the "closeness" condition

(C) There exists a continuous function $\lambda : [0, \infty) \rightarrow [0, \infty)$ such that

$$
|F(t, \varphi) - g(t, \varphi(0))| \leq \lambda(t) \|\varphi'\| \tag{1.1}
$$

for any continuously differentiable function $\varphi : [-q, 0] \rightarrow \mathbb{R}^n$.

We Remark that (1.1) holds with $\|\varphi'\|$ and not with $\|\varphi\|$. See [15].

We wish to study the relation between the solutions of the functional differential system
and the solutions of the ordinary differential system

\[ x'(t) = g(t, x(t)) \]  

For system (1.3) we suppose that the following condition is fulfilled:

(G) The derivative of \( g \) : \( g_x = g_x(t, x) \) exists and is continuous on \([0, \infty) \times \mathbb{R}^n\). System (1.3) is an h-system in variation with radius of attraction \( \delta \), where \( h : [0, \infty) \to (0, \infty) \) is a continuous function.

We recall that a system (1.3) or its null-solution is an h-system in variation [5, 6] with radius of attraction \( \delta \), if there exist a continuous function \( h : [0, \infty) \to (0, \infty) \) and constants \( K \geq 1 \) and \( \delta > 0 \) such that for \( 0 < |x_0| < \delta \) we have

\[ |\Phi(t, t_0, x_0)| \leq K h(t) h(t_0)^{-1} \quad (t \geq t_0 \geq 0), \]

where \( \Phi(t, t_0, x_0) \) is the fundamental matrix of the variational system

\[ z'(t) = g_x(t, z(t, t_0, x_0)) z(t) \]

such that \( \Phi(t, t_0, x_0) = Id \) (the identity matrix). Here \( x = x(t, t_0, x_0) \) represents the solution \( x \) passing through the point \((t_0, x_0)\).

This problem appears in Bellman [1] who proposed to investigate conditions on the lag \( r \) to know the behavior of solutions of the functional differential equation

\[ u'(t) + au(t - r(t)) = 0, \quad a \text{ constant} \]  

when \( r(t) \to 0 \) as \( t \to \infty \). In [2], Cooke proves that if \( a > 0 \) and \( r \in L_1([0, \infty)) \) then any solution \( u \) of (1.1) satisfies

\[ u(t) = e^{-at}[c + o(1)], \quad t \to \infty \]

for some constant \( c \). In [3], Cooke generalizes this result to linear systems of functional differential equations asymptotically autonomous. Grossman and Yorke [4] consider the one-dimensional functional differential equation

\[ u'(t) = a(t)u(t - r(t)). \]

In [10] we have extended some of these results to the scalar functional equation

\[ u'(t) = a(t)u(t - r(t, u)) \]
with a lag of implicit type, generalizing the case

\[ u'(t) = -au(t - r(u(t))) \]

studied by Cooke [4]. See also [12, 14]. We note that in all of these cases the estimate (1.1) does not hold with \( \| \varphi' \| \) instead of \( \| \varphi \| \).

In this paper, for the nonlinear problem (1.2), we obtain the relation

\[ y = x + h \cdot \hat{\delta}(1), \]

between the solutions \( y \) of (1.2) and \( x \) of (1.3), where \( \hat{\delta}(1) \) is a convergent function as \( t \to \infty \). We will prove also that the nonlinear functional system (1.2) is an \( h \)-system (see Remark 1).

As an application we get asymptotic formulae of the solutions of second order delay equation [11, 13]

\[ y'' + c(t)y(t - r(t)) = 0 \]

in terms of the solutions of

\[ z'' + c(t)z = 0, \]

extending ordinary results [7, 8].

2 Main Results

In this section we get asymptotic formulae for the solutions of system (1.2). We denote by \( y = y(t; t_0, y_{t_0}) \) a solution \( y \) of Eq. (1.2) with initial function \( y_{t_0} \in C_0 \).

**Theorem 1** In addition to conditions \( C \) and \( G \), assume:

(i) There exists a continuous and nonnegative function \( c(t) \) such that

\[ |F(t, \varphi)| \leq c(t)\| \varphi \| \]

for all \( t \geq 0 \) and all \( \varphi \in C_0 \).

(ii) \( \beta(t)\lambda(t)\| c_t \| \in L_1([0, \infty)) \), where \( \beta(t) = h(t)^{-1}\| h_t \|_2 \).

Then for any solution \( y = y(t; t_0, y_{t_0}) \) of (1.2) with \( \| y_{t_0} \| \leq \delta \) there exists a solution \( x \) of (1.3) such that

\[ y = x + h \cdot \hat{\delta}(1), \]

where \( \hat{\delta}(1) \) is a function defined on \([t_0, \infty)\) which converges as \( t \to \infty \).
Proof. By condition (G), for \( |y(t_0)| \leq \delta \), the solution \( x = x(t; t_0, y(t_0)) \) of the ordinary system (1.3) is well defined and satisfies \( |x(t; t_0, y(t_0))| \leq K|y(t_0)|h(t_0)h(t_0)^{-1} \) for \( t \geq t_0 \geq 0 \) and \( K \geq 1 \) a constant. Now, by (i), the solution \( y = y(t, t_0, y_{t_0}) \) of system (1.2) is defined on \([t_0 - q, \infty)\). By the formula of variation of the constants, we have for \( t \geq t_1 \geq t_0 \)

\[
y(t) = x(t; t_1, y(t_1)) + \int_{t_1}^{t} \Phi(t, s, y(s))[F(s, y_s) - g(s, y(s))]ds \tag{2.1}
\]

Then, by (C) and (G)

\[
|y(t)| \leq K|y(t_1)|h(t_1)h(t_1)^{-1} + K h(t) \int_{t_1}^{t} h(s)^{-1} \lambda(s)\|y_s\|ds
\]
or

\[
h(t)^{-1}|y(t)| \leq K h(t_1)^{-1}|y(t_1)| + K \int_{t_1}^{t} \lambda(s)h(s)^{-1}\|y_s\|ds.
\]

Thus \( z(t) = h(t)^{-1}|y(t)| \) satisfies

\[
z(t) \leq K z(t_1) + \int_{t_1}^{t} K \lambda(s)h(s)^{-1}\|y_s\|ds \tag{2.2}
\]

For \( u \in [-q, 0] \) and \( s \geq t_1 \), by (i), we have

\[
|y'(u)| = |F(s + u, y_{s+u})| \leq c_s(u)\|y_{s+u}\| = c_s(u)|y(v)|
\]

for some \( v = v(s) \in [s - 2q, s] \). Further

\[
h(s)^{-1}|y(v)| = h(s)^{-1}h(v)z(v) \leq \beta(s)z(v).
\]

Thus

\[
h(s)^{-1}\|y_s\| \leq \beta(s)c_s\|m(s)\|, \tag{2.3}
\]

where \( m(t) = \max_{-2q \leq s \leq t} |z(s)| \). Substituting this into (2.2) we obtain

\[
z(t) \leq K z(t_1) + \int_{t_1}^{t} K \lambda(s)\|y_s\|ds. \tag{2.4}
\]

Since the right member of (2.4) is increasing as a function in \( t \), for \( t \geq t_1 + 2q \) we have \( m(t) \leq K z(t_1) + \int_{t_1}^{t} K r(s)\|y_s\|m(s)ds \). Then by (ii), Gronwall’s inequality implies that \( m \) and hence \( z \) are bounded. Moreover, for any \( t \) fixed \( \Phi(t, s, y(s))[F(s, y_s) - g(s, y(s))] \in L_1([0, \infty)) \) as a function of \( s \) because from (C), (G), (ii) and (2.3) we get

\[
|\Phi(t, s, y(s))[F(s, y_s) - g(s, y(s))]| \leq K h(t)h(s)^{-1} \lambda(s)\|y_s\|
\]

\[
\leq K_1 h(t)\|c_s\|\lambda(s)\|\beta(s)m(s) \leq K_2 h(t)\|\lambda(s)\|\beta(s)c_s \|m(s) \in L_1([0, \infty)).
\]

Then the integral in (2.1) can be written as \( h(t) \cdot \tilde{\alpha}(1) \), where \( \tilde{\alpha}(1) \) denotes a function of \( t \) which has a limit as \( t \to \infty \). The proof is complete.
Remark 1. Since we have proved $h(t)^{-1}|y(t)| \leq m(t) \leq KK_1 |z(t_1)| = KK_1 h(t_1)^{-1}|y(t_1)|$ for $t \geq t_1 \geq t_0$ and $K_1$ a positive constant, we have also established

$$|y(t)| \leq K_2 h(t)^{-1} h(t_1)^{-1}|y(t_1)|, \quad (t \geq t_1 \geq t_0), \quad K_2 \text{ constant}$$

that is, the nonlinear functional system (1.2) is also an h-system.

Theorem 1 includes the interesting type of equations as:

$$y' = F(t, y(t) - y(t - r(t))), \quad (2.5)$$

where $r : [0, \infty) \to [0, q]$ is a continuous function.

For this equation, system (1.3) becomes $x' = 0$ and (1.1) becomes

$$|F(t, \varphi)| \leq r(t)\|\varphi\| \quad (2.6)$$

Thus here $h \equiv 1, \quad \beta \equiv 1 \quad \text{and we have}$

**Corollary 1** Assume that (i) of Theorem 1 and (2.6) hold. If $r(t) \cdot \|c_i\| \in L_1([0, \infty))$, then any solution $y = y(t; t_0, y_0)$ of (2.5) there exists a constant vector $v$ such that

$$y = y(t_0) + v + o(1)$$

as $t \to \infty$. In particular, any solution of (2.5) is asymptotically constant.

Proceeding as in the proof of the Theorem 1, with a Bihari's inequality, Corollary 1 can be obtained for the nonlinear equations

$$y' = y^3(t) - y^3(t - r(t)) \quad \text{or} \quad y' = [y(t) - y(t - r(t))]^3$$

since in this case we have an estimate of the type:

$$|F(t, y)| \leq K r(t) w(||y'||), \quad (2.7)$$

where $w : (0, \infty) \to (0, \infty)$ is a continuous, nondecreasing function satisfying $w(0) \geq 0$ and

$$\int_{0^+}^{1} \frac{ds}{w(s)} = \infty \quad (2.8)$$

Thus from lemma 1, [6] we obtain:

**Corollary 2** Under the conditions of Corollary 1 with (2.7-2.8) instead of (2.6), there exists a constant $\rho > 0$ such that any solution $y = y(t; t_0, y_0)$ with $\|y_0\| \leq \rho$ is defined on $[t_0 - q, \infty)$ and

$$y = y(t_0) + v(t_0) + o(1), \quad t \to \infty \quad (2.9)$$
where \( v = v(t_0) \) is a constant vector such that \( v(t_0) \to 0 \) as \( t_0 \to \infty \). Moreover, \( \rho = \rho(t_0) \) verifies \( \rho(t_0) \to \infty \) as \( t_0 \to \infty \). Then if \( t_0 \) is chosen large enough for any initial function \( \varphi \) there exists \( t_0 \) large enough such that the solution \( y = y(t, t_0, \varphi) \) verifies the above asymptotic formulae.

Some simple consequences are the following:

**Corollary 3** If, for \( h(t) = \exp(\int_0^t a(s)ds) \), \( a||a_i||h(t)^{-1}||h^t||2r \in L_1([0, \infty)) \), then the solutions of the scalar equation

\[
y'(t) = a(t)y(t - r(t)),
\]

satisfy

\[
y(t) = \exp(\int_0^t a(s)ds)[c + o(1)], \quad c \text{ constant}.
\]

Thus, in particular, the solutions of

\[
y'(t) = -ty(t - e^{-3t})
\]

and

\[
y'(t) = ty(t - r(t)), t^2r(t) \in L_1([0, \infty)),
\]

satisfy respectively,

\[
y = e^{-t^2/2}[c + o(1)], \quad c \text{ constant}
\]

and

\[
y = e^t[c + o(1)], \quad c \text{ constant}.
\]

Now, an explicit nonlinear scalar example is shown. Let \( g(t, x) = -e^tx^3 \) in equation (1.3):

\[
x'(t) = -e^tx^3(t)
\]

This ordinary system has the solutions

\[
x(t, t_0, x_0) = \frac{|x_0|}{(1 + 2x_0^2(e^t - e^{t_0}))^{1/2}}
\]

whence it is an h-system with \( h(t) = e^{-t/2} \). Then, Theorem 1 implies that the solutions \( y = y(t, t_0, y_{t_0}) \) of the scalar equation

\[
y'(t) = -e^ty^3(t - e^{-\alpha t}), \alpha > 2,
\]

satisfy

\[
y(t) = x(t) + e^{-t/2} \cdot o(1),
\]

for \( t \) large enough.
Corollary 4 If $A$ is a stable matrix, then any solution of
\[ y' = Ay(t-r(t)), \quad r \in L_1([0,\infty)) \]
satisfies
\[ y = e^{tA}x_0 + e^{-\alpha t} \cdot \delta(1) \]
where $x_0$ is a constant vector, $0 > \alpha > \max \Re \lambda$ for $\lambda$ an eigenvalue of $A$ and $\delta(1)$ is a convergent vector as $t \to \infty$.

When (1.3) is a linear and an h-system (see [6]) we have:

Corollary 5 If system
\[ x' = A(t)x \quad (2.10) \]
is an h-system and $rh(t)^{-1} ||h'||_2 ||A|| ||A_t|| \in L_1([0,\infty))$, then for any solution $y$ of
\[ y' = A(t)y(t-r(t)) \]
satisfies
\[ y = \Phi y_0 + h\delta(1) \quad \text{as} \quad t \to \infty \]
where $y_0$ is a constant vector and $\Phi$ is a fundamental matrix of (2.10).

3 An application: Asymptotic formulae for the solutions of (1.5)

Consider the functional differential equation
\[ y'' + c(t)y(t-r(t)) = 0 \quad (3.1) \]
where $c : [0,\infty) \to \mathbb{R}$ and $r : [0,\infty) \to [0,q]$ are continuous functions.

As usually, a solution of eq. (3.1) is a function $y = y(t;t_0,\varphi,\psi)$ such that $y$ satisfies the delay-differential equations (3.1) and
\[ y_{t_0} = \varphi, \quad y'_{t_0} = \psi, \]
where $\varphi, \psi \in C([-q,0], \mathbb{R})$.

For $r = r(t)$ small, in some sense which will be precised, we hope that the solutions $y$ of Eq (3.1) behave asymptotically as the solutions $z$ of the ordinary differential equation
\[ z'' + c(t)z(t) = 0. \]  

(3.2)

We will prove that any solution \( y \) of Eq (3.1) are defined on all of \( I = [0, \infty) \) and it satisfies as \( t \to \infty \):

\[ y = (\delta_1 + o(1))z_1 + (\delta_2 + o(1))z_2 \]  

(3.3)

\[ y' = (\delta_1 + o(1))z_1' + (\delta_2 + o(1))z_2' \]

where \( \{z_1, z_2\} \) is a fundamental system of solutions of Eq (3.2) and \( \{\delta_1, \delta_2\} \) are constants. Let

\[ y(t) = A(t)z_1(t) + B(t)z_2(t) \]  

(3.4)

under the condition

\[ A'z_1 + B'z_2 = 0 \]  

(3.5)

Then, we have \( y' = Az_1' + Bz_2' \) and \( y'' = A'z_1' + B'z_2' + Az_1'' + Bz_2'' \). Thus \( y'' = A'z_1' + B'z_2' - c(Az_1 + Bz_2) \). Therefore

\[ A'z_1' + B'z_2' = c(t)[y(t) - y(t - r(t))]. \]  

(3.6)

Solving Eqs. (3.5) and (3.6), we get

\[ A' = -w^{-1}z_1 \cdot c(t)[y(t) - y(t - r(t))] \]  

(3.7)

\[ B' = w^{-1}z_1 \cdot c(t)[y(t) - y(t - r(t))] \]

where \( w \) is the Wronskian of system \( \{z_1, z_2\} \). Now, we have

\[ |y(t) - y(t - r(t))| = \left| \int_{t-r(t)}^{t} y'(s)ds \right| = \left| \int_{t-r(t)}^{0} y'(t + s)ds \right| \]

\[ = \left| \int_{0}^{t-r(t)} y'(s)ds \right| = \left| \int_{t-r(t)}^{0} (Az_1' + Bz_2')ds \right|. \]

Thus

\[ |y(t) - y(t - r(t))| \leq r(t) \max_{i=1,2} \|z_i'\| \cdot (\|A_i\| + \|B_i\|). \]

Then, by system (3.7), the vector \( x = (A, B) \) satisfies a system of functional differential equations of the type

\[ x' = F(t, x_t) \]  

(3.8)

satisfying the conditions (i) \( F : I \times C_0 \to \mathbb{R} \) is a continuous function (ii) \( |F(t, \varphi)| \leq \lambda(t)\|\varphi\|, \ (t, \varphi) \in I \times C_0. \)
In this point, we need the following Theorem concerning the asymptotic behavior of system (3.8).

**Theorem 2** Assume the above conditions (i) and (ii), where $\lambda \in C(I, \mathbb{R})$ satisfies $\lambda(t) \in L_1(I)$. Then the solutions with continuous initial conditions of Eq (3.8) are defined on all of $I$ and they converge as $t \to \infty$.

The proof of this theorem is omitted because it is similar to that of Theorem 1.

Thus, we get:

**Theorem 3** Assume that $r(t)|c(t)| \cdot |z_i(t)| \cdot \|z_i^\prime\| \in L_1(I) \quad i = 1, 2$. Then any solution $y = y(t; t_0, y_{t_0}, y_{t_0}')$ satisfies formulae (3.3).

**Proof.** The application of Theorem 2 implies that $A$ and $B$ converge as $t \to \infty$. The formulae (3.3) follow from (3.4) and $y' = Az_i' + Bz_i^\prime$.

So, we have

**Corollary 6** If $r \in L_1(I)$, then any solution $y$ of the functional differential equation

$$y'' + ay(t - r(t)) = 0, \quad a > 0 \quad \text{constant}$$

satisfies for $t \to \infty$,

$$y = (\delta_1 + o(1)) \sin at + (\delta_2 + o(1)) \cos at$$

$$y' = a(\delta_1 + o(1)) \cos at - a(\delta_2 + o(1)) \sin at$$

More generally, using Green-Liouville formulae ([7]) for the solutions of (3.2) we get:

**Corollary 7** If $c(t) \in C^2(I), c > 0$ and $c^{-3/2}c''$, $r(t) \cdot |c^{3/4}(t)||c_{3/4}^\prime| \in L_1(I)$ then any solution $y$ of the functional differential equation

$$y'' + c(t)y(t - r(t)) = 0$$

satisfies for $t \to \infty$

$$y = c(t)^{-1/4}[(\delta_1 + o(1))\exp(i \int c^{1/2}(s)ds) + (\delta_2 + o(1))\exp(-i \int c^{1/2}(s)ds)]$$

$$y' = c(t)^{1/4}[i(\delta_1 + o(1))\exp(i \int c^{1/2}(s)ds) + i(\delta_2 + o(1))\exp(-i \int c^{1/2}(s)ds)]$$

For more related results, see [9].

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REFERENCES


11. BURTON, T. A. & HADDOCK, J. R. On the delay-differential equations $x'(t) + a(t)f(x(t - r(t))) = 0$ and $x''(t) + a(t)f(x(t - r(t))) = 0$, J. Math. Anal. Appl. 54 (1976), 37-48.


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