ON A CONJECTURE OF VUKMAN

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ABSTRACT. Let R be a ring. A bi-additive symmetric mapping \( d : R \times R \to R \) is called a symmetric bi-derivation if, for any fixed \( y \in R \), the mapping \( x \to D(x, y) \) is a derivation. The purpose of this paper is to prove the following conjecture of Vukman:

Let \( R \) be a noncommutative prime ring with suitable characteristic restrictions, and let \( D : R \times R \to R \) and \( f(z) = D(z, z) \) be a symmetric bi-derivation and its trace, respectively. Suppose that \( f(x) \in Z(R) \) for all \( x \in R \), where \( f_{k+1}(x) = [f_k(x), x] \) for \( k \geq 1 \) and \( f_1(x) = f(x) \), then \( D = 0 \).

KEY WORDS AND PHRASES: Prime ring, centralizing mapping, symmetric bi-derivation.

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1. INTRODUCTION

Throughout this paper, \( R \) will denote an associative ring with center \( Z(R) \). We write \([x, y]\) for \( xy - yx \), and \( I_a \) for the inner derivation deduced by a \( a \). A mapping \( D : R \times R \to R \) will be called symmetric if \( D(x, y) \) holds for all pairs \( x, y \in R \). A symmetric mapping is called a symmetric bi-derivation, if \( D(x + y, z) = D(x, z) + D(y, z) \) and \( D(xy, z) = D(x, z)y + xD(y, z) \) are fulfilled for all \( x, y \in R \). The mapping \( f : R \to R \) defined by \( f(x) = D(x, x) \) is called the trace of the symmetric bi-derivation \( D \), and obviously, \( f(x + y) = f(x) + f(y) + 2D(x, y) \). The concept of a symmetric bi-derivation was introduced by Gy Maksa in [1,2]. Some recent results concerning symmetric bi-derivations of prime rings can be found in Vukman [3,4]. In [4], Vukman proved that there are no nonzero symmetric bi-derivations \( D \) in a noncommutative prime ring \( R \) of characteristic not two and three, such that \( [[D(x, x), x], x] \in Z(R) \). The following conjecture was raised: Let \( R \) be a noncommutative prime ring of characteristic different from two and three, and let \( D : R \times R \to R \) be a symmetric bi-derivation. Suppose that for some integer \( n \geq 1 \), we have \( f_n(x) \in Z(R) \) for all \( x \in R \), where \( f_{k+1}(x) = [f_k(x), x] \) for \( k = 1, 2, \ldots, \) and \( f_1(x) = D(x, x) \). Then \( D = 0 \).

The purpose of this paper is to prove this conjecture under suitable characteristic restrictions.

2. THE RESULTS

THEOREM 1. Let \( R \) be a prime ring of characteristic different from two. Suppose that \( R \) admits a nonzero symmetric bi-derivation. Then \( R \) contains no zero divisors.

PROOF. It is sufficient to show that, \( a^2 = 0 \) for \( a \in R \) implies \( a = 0 \). We need three steps to establish this.

LEMMA A. If \( D(a, *) \neq 0 \), then \( D(a, *) = \mu I_a \), where \( \mu \in C \), the extended centroid of \( R \).

PROOF. Since \( D(a^2, x) = D(0, x) = 0 \), we have
Replacing $x$ by $xy$, we obtain

$$I_a(x)D(a,y) = D(a,x)I_a(y)$$

for all $x \in R$.

and replacing $y$ by $yz$, we get

$$I_a(x)yD(a,z) = D(a,x)yI_a(z)$$

(2.1)

Since $D(a,*) \neq 0$, we may suppose that $D(a,z) \neq 0$ for a fixed $z \in R$. Obviously $I_a(z) \neq 0$. By (2.1), and by [5, Lemma 1.3.2], there exist $\mu(x)$ and $\nu(x)$ in $C$, either $\mu(x)$ or $\nu(x)$ being not zero, such that $\mu(x)I_a(x) + \nu(x)D(a,x) = 0$. If $\nu(x) \neq 0$ then $D(a,x) = \frac{\mu(x)}{\nu(x)} I_a(x)$; on the other hand, if $\nu(x) = 0$ then $\mu(x)I_a(x) = 0$ and $I_a(x) = 0$, using (2.1) and $I_a(z) \neq 0$, so $D(a,x) = 0$. In any event, we have $D(a,x) = \mu(x)I_a(x)$

Hence (2.1) implies $(\mu(x) - \mu(z))I_a(z) = 0$. It follows that either $I_a(z) = 0$ or $\mu(x) = \mu(z)$. By (2.1), the former implies $D(a,x) = 0$ and $D(a,x) = \mu(z)I_a(x)$.

In both cases, we get $D(a,x) = \mu(z)I_a(x)$ for all $x \in R$, and $\neq \mu(z)$ being fixed.

The fixed element $\mu$ in Lemma A is somewhat dependent on $a$, we write it as $\mu_a$. For any given $r \in R$, $ara$ satisfies our original hypotheses on $a$; therefore for each $r \in R$, either $D(ara,*) = 0$ or $D(ara,*) = \mu_aI_{ara}$, where $\mu_{ara} \neq 0$.

**LEMMA B.** If $D(ara,*) \neq 0$, then $\mu_{ara} = \mu_a$.

**PROOF.** $D(ara,*) \neq 0$ implies $ara \neq 0$. Suppose that $D(a,*) = 0$, then $D(ara,x) = D(a,x)ra + aD(r,x)a = \mu_aI_{ara},$ but $D(ara,x) = \mu_{ara}I_{ara}(x) = \mu_a(ara - ara)$, so that $\mu_{ara}(ara - ara) = D(r,x)a.$ Right-multiplying the last equation by $a$, we have $\mu_{ara}ara = 0$ for all $x \in R$. It follows that $ara = 0$, a contradiction. Therefore $D(a,*) = \mu_aI_a$, and consequently,

$$D(ara,x) = \mu_aI_a(x)ra + aD(r,x)a + ar\mu_a(x);$$

and right-multiplying this equation by $a$ yields

$$D(ara,x)a = \mu_aara$$

for all $x \in R$.

Hence $\mu_{ara}ara = \mu_aara$, immediately $\mu_{ara} = \mu_a$.

**LEMMA C.** If $a^2 = 0$, then $a = 0$.

**PROOF.** Let $S = \{r \in R | D(ara,*) = \mu_aI_{ara}, \mu_{ara} \neq 0\}$ and $T = \{r \in R \setminus D(ara,*) = 0\}$.

By Lemma A and B, $R = S \cup T$ and $S$ and $T$ are additive subgroups of $R$. We conclude that either $S = R$ or $T = R$.

Suppose that $S = R$. Lemma A gives, either $D(a,*) = 0$ or $D(a,*) = \mu_aI_a$. If $D(a,*) = 0$, then $D(ara,x) = D(r,x)a$, for all $r,x \in R$, and $D(ara,x)a = 0$. It follows that $\mu_{ara} = 0$. Since $\mu_a = \mu_{ara} \neq 0$, we have $a = 0$. If $D(a,*) = \mu_aI_a$, then the equation

$$D(ara, ya) = D(a, ya)ra + aD(r,y)a + arD(a,y)a$$

gives $\mu_{ara}ya = 2\mu_aayara + \mu_{ara}ya$. Hence we get $ayara = 0$, and $a = 0$ again.

We suppose henceforth that $T = R$. If $D(a,*) = 0$, then $D(aza,yz) = aD(aza,yz) = 0$, and $ayD(za,z) = 0$. Thus $D(za,z) = D(z)za = 0$. Since $D \neq 0$, we then get $a = 0$. If $D(a,*) = \mu_aI_a$, then, right-multiplying the equation $D(aza,y) = 0$ by $a$, we obtain $\mu_aaza = azD(a,y)a = 0$, and $a = 0$ again. The proof of the theorem is complete.

In order to prove Vukman’s conjecture, we need the following proposition.

**PROPOSITION.** Let $n$ be a positive integer; let $R$ be a prime ring with char $R = 0$ or char $R > n$; and let $g$ be a derivation of $R$ and $f$ the trace of a symmetric bi-derivation $D$. For $i = 1, 2, ..., n$, let $F_i(X,Y,Z)$ be a generalized polynomial such that $F_i(kx, f(kz), g(kz)) = k^iF_i(x, f(x), g(x))$ for all $x \in R$ for $k = 1, 2, ..., n$. Let $a \in R$, and (a) the additive subgroup generated by $a$. If all $x \in (a)$,
\[ F_n(x, f(x), g(x)) + F_{n-1}(x, f(x), g(x)) + \ldots + F(x, f(x), g(x)) \in Z(R), \]  
(2.2)

then \( F_i(a, f(a), g(a)) \in Z(R) \) for \( i = 1, 2, \ldots, n \).

This proposition can be proved by replacing \( x \) by \( a, 2a, \ldots, na \) in (2.2) and applying a standard "Van der Monde argument".

**Theorem 2.** Let \( n \) be a fixed positive integer and \( R \) be a prime ring with \( \text{char } R = 0 \) or \( \text{char } R > n + 2 \). Let \( f_{k+1}(x) = [f_k(x), x] \) for \( k > 1 \), and \( f_1(x) = f(x) \) the trace of a symmetric bi-derivation \( D \) of \( R \). If \( f_n(x) \in Z(R) \) for all \( x \in R \), then either \( D = 0 \) or \( R \) is commutative.

**Proof.** Linearizing \( f_n(x) \in Z(R) \), we obtain

\[ \ldots ([f(x), y] + f(x), x, x + y], x + y] \in Z(R); \]

and using the Proposition, we get

\[ \ldots ([f(x), y], x, \ldots, x + y], x + y] + 2\ldots ([D(x, y), x], x, \ldots, x] \in Z(R), \]
equivalently,

\[
(-1)^{n-2}I_{x}^{n-2}([f_1(x), y]) + (-1)^{n-3}I_{x}^{n-3}([f_2(x), y]) + \ldots + [f_{n-1}(x), y] + 2(-1)^{n-1}I_{x}^{n-1}(D(x, y)) \in Z(R). \quad (2.3)
\]

Noting that

\[
(-1)^{-n-2}I_{x}^{n-2}([f_1(x), x^2]) = (-1)^{-n}([f_2(x), x^2]) = \ldots = [f_{n-1}(x), x^2] = (-1)^{-n-1}I_{x}^{n-1}(D(x, x^2)) = 2f_n(x)x,
\]

and replacing \( y \) by \( x^2 \) in (2.3), we then get \( 2(n + 1)f_n(x)x \in Z(R) \). Since \( f_n(x) \in Z(R) \), it follows that \( f_n(x) = 0 \).

The linearization of \( f_n(x) \) gives

\[
(-1)^{n-2}I_{x}^{n-1}([f_1(x), y]) + (-1)^{n-3}I_{x}^{n-3}([f_2(x), y]) + \ldots + [f_{n-1}(x), y] + 2(-1)^{n-1}I_{x}^{n-1}(D(x, y)) = 0. \quad (2.4)
\]

Since \( I_{x}^{n-k}([f_{k-1}(x), xy]) = xI_{x}^{n-1}([f_{k-1}(x), y]) + I_{x}^{n-k}([f_k(x), y]) \) for \( k = 2, 3, \ldots, n \), and \( I_{x}^{n-1}(D(x, xy)) = xI_{x}^{n-1}(D(x, y)) + I_{x}^{n-1}([f_1(x), y]) \). Substituting \( xy \) for \( y \) in (2.4), we have

\[
(-1)^{n-2}I_{x}^{n-2}(f_2(x)y) + (-1)^{n-3}I_{x}^{n-3}(f_3(x)y) + \ldots + (-1)^{n-1}I_{x}^{n-1}(f_1(x)y) = 0.
\]

Taking \( y = f_{n-2}(x) \), applying \( I_{x}^{n}((ab) = k \sum_{j=0}^{k} \binom{k}{j} I_{x}^{k-j}(a)I_{x}^{j}(b) \) and noting \( I_{x}^{n}(f(x)) = 0 \) for \( i \) and \( j \), we then conclude that

\[
2(-1)^{n-1} \left( \binom{n-1}{1} \right) I_{x}^{n-2}(f_1(x)I_{x}(f_{n-2}(x))) + (-1)^{n-2} \left( \binom{n-2}{1} \right) I_{x}^{n-3}(f_2(x))I_{x}(f_{n-2}(x)) + \ldots + (-1)^{n-1}f_{n-1}(x)I_{x}(f_{n-2}(x)) = 0.
\]

But \( (-1)^{k}I_{x}^{k-1}(f_{n-k}(x))I_{x}(f_{n-2}(x)) = (f_{n-1}(x))^2 \), so \( (n + 2)(n - 1)(f_{n-1}(x))^2 = 0 \), and by the hypotheses on the characteristic, we get \( (f_{n-1}(x))^2 = 0 \). Suppose that \( D \neq 0 \). By Theorem 1, \( f_{n-1}(x) = 0 \), and by induction, \( f_2(x) = [f(x), x] = 0 \). Using Vukman [3, Theorem 1], \( R \) is commutative, we complete the proof of Theorem 2.

**Theorem 3.** Let \( n > 1 \) be an integer and \( R \) be a prime ring with \( \text{char } R = 0 \) or \( \text{char } R > n + 1 \), and let \( f(x) \) be the trace of a symmetric bi-derivation \( D \) of \( R \). Suppose that \( [x^2, f(x)] \in Z(R) \) for all \( x \in R \). In this case either \( D = 0 \) or \( R \) is commutative.
PROOF. Using the condition \([x^n, f(x)] \in Z(R)\), we get \([x^{2n}, f(x^2)] \in Z(R)\), and
\[
[x^{2n}, f(x)]x^2 + x^2[x^{2n}, f(x)] + 2x[x^{2n}, f(x)]x \in Z(R).
\]
(2.5)

Noting that \([x^{2n}, f(x)] = 2[x^n, f(x)]x^n\), we now have from (2.5) that \(8[x^n, f(x)]x^{n+2} \in Z(R)\). Thus either \([x^n, f(x)] = 0 \text{ or } x^{n+2} \in Z(R)\).

But linearizing \([x^n, f(x)] \in Z(R)\) and applying the Proposition gives
\[
[x^{n-1}y + x^{n-2}yx + ... + yx^{n-1}, f(x)] + 2[x^n, D(x, y)] \in Z(R)
\]
for all \(x, y \in R\), and taking \(y = x^3\), yields
\[
n[x^{n+2}, f(x)] + 6[x^n, f(x)]x^2 \in Z(R).
\]

Suppose that \([x^n, f(x)] \neq 0\), then \(x^{n+2} \in Z(R)\) and \([x^n, f(x)]x^2 \in Z(R)\), hence \(x^2 \in Z(R)\). Now this condition, together with \(x^{n+2} \in Z(R)\), implies either \(x^2 = 0\) or \(x^n \in Z(R)\), so that in each event, \([x^n, f(x)] = 0\).

Linearizing \([x^n, f(x)] = 0\) and using the Proposition, we have
\[
[x^{n-1}y + x^{n-2}yx + ... + yx^{n-1}, f(x)] + 2[x^n, D(x, y)] = 0
\]
Replacing \(y\) by \(x^2\) yields \(n[x^{n+1}, f(x)] = 0\), hence \([x, f(x)]x^n = 0\). If \(D \neq 0\), then by Theorem 1, \([x, f(x)] = 0\), and by Vukman [3, Theorem 1], \(R\) is commutative. This completes the proof.

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REFERENCES


