HAMILTONIAN-CONNECTED GRAPHS AND THEIR STRONG CLOSURES

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ABSTRACT. Let G be a simple graph of order at least three. We show that G is Hamiltonian-connected if and only if its strong closure is Hamiltonian-connected. We also give an efficient algorithm to compute the strong closure of G.

KEY WORDS AND PHRASES. Hamiltonian-connected graph, strong closure, degree sequence.

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1. INTRODUCTION.

Let G = (V, E) be a simple graph, n = |V| (≥ 3) and m = |E|. G is called Hamiltonian-connected if every two vertices of G are connected by a Hamiltonian path. If G is Hamiltonian-connected and n ≥ 4, then m ≥ 1/2 (3n + 1) (see [2], p. 61).

In this paper, we define the strong closure sc(G) of a simple graph G. We also show that G is Hamiltonian-connected if and only if its strong closure sc(G) is Hamiltonian-connected (Theorem 2.3). It follows immediately that if sc(G) is a complete graph, then G is Hamiltonian-connected (Corollary 2.4). As in the case of Hamiltonian graphs, there is no characterization of Hamiltonian-connected graphs. If we compute the strong closure of G and find it is complete, then G is Hamiltonian-connected. As another application, a result of O. Ore also follows from Corollary 2.4 (see Corollary 2.5).

In section 3, we give an efficient algorithm to compute the strong closure sc(G) of any simple graph G. This algorithm can be executed in O(n |K|) time, where |K| = 1/2 (n^2 - n - 2m).

2. HAMILTONIAN-CONNECTED GRAPHS.

For each vertex v of G, let D(v) = {u ∈ V(G): u is adjacent to v}. Then d(v) = |D(v)| is the degree of v in G.

We have the main result of this paper.

THEOREM 2.1. Suppose that u is not adjacent to v in G and d(u) + d(v) ≥ n + 1. Then G is Hamiltonian-connected if and only if G + (u, v) is Hamiltonian-connected.

PROOF. Suppose that G + (u, v) is Hamiltonian-connected, but G is not. Since G is not Hamiltonian-connected, there exist two vertices x and y such that there is no Hamiltonian x - y path in G. Since G + (u, v) is Hamiltonian-connected, there is a Hamiltonian u - v path in G + (u, v) and hence in G. Therefore it follows that (x, y) ≠ (u, v). Let P = {w_1, w_2, ..., w_n} be a Hamiltonian x - y path in G + (u, v), where x = w_1 and y = w_n.
CASE 1. Assume that $x \neq u$ and $y \neq v$. Since $P$ is a Hamiltonian $x-y$ path in $G+(u,v)$ and $P$ is not a Hamiltonian $x-y$ path in $G$, $(u,v)$ must be an edge of $P$ in $G+(u,v)$. Therefore $u = w_k$ and $v = w_{k+1}$ for some $1 < k < n-1$. ($k \neq n-1$; for otherwise $k+1 = n$ and $v = w_n = y$). Since $(u,v)$ is not an edge of $G$, $u \notin D(u)$ and $v \notin D(v)$. Suppose $w_t \in D(u)$, where $t \neq k-1$ and $t \neq n$. Since $w_t \in D(u)$ and $v \notin D(u)$ are not in $D(u)$, it follows that $t \neq k$ and $t \neq k+1$. We show that $w_{t+1} \notin D(v)$. Suppose that this is not true, then $w_{t+1}$ is adjacent to $v$. If $t < k-1$, then the path $(=w_1), w_2, \ldots , w_t, w_k (=u), w_{k-1}, w_{k-2}, \ldots , w_{t+1}, v(=w_{k+1}), w_{k+2}, \ldots , y(=w_n)$ is a Hamiltonian $x-y$ path in $G$. If $t > k+1$, then the path $x(=w_1), w_2, \ldots , w_k (=u), w_{t}, w_{t-1}, \ldots , w_{k+1}(=v), w_{t+1}, w_{t+2}, \ldots , y(=w_n)$ is a Hamiltonian $x-y$ path in $G$. This is impossible. Therefore, $w_{t+1} \notin D(v)$. Since $t \neq k-1$ and $t \neq n$, it follows that there are at least $d(u)-2$ vertices to which $v$ is not adjacent. Since $u,v \notin D(v)$, we have $d(v) \leq (n-2) - (d(u)-2) = n - d(u)$.

Therefore $d(u) + d(v) \leq n$, which is a contradiction. Therefore Case 1 is impossible.

CASE 2. Assume that $v = y(=w_n)$. Since $(x,y) \neq (u,v)$, it follows that $u \neq x$ and so $u = w_{n-1}$. Let $w_t \in D(u)$, where $t \neq n-2$. Then by the same argument as in Case 1, $w_{t+1} \notin D(v)$. Hence $d(v) \leq (n-2) - (d(u)-1) = n - d(u) - 1$

and so $d(u) + d(v) \leq n-1$, which is impossible. Therefore $G$ is Hamiltonian-connected. The converse of the theorem is clearly true. This completes the proof of the theorem.

Theorem 2.1 motivates the following definition.

The strong closure of $G$ is the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $n+1$ until no such pair remains. We denote the strong closure of $G$ by $sc(G)$.

**REMARK.** The closure $c(G)$ of $G$ is defined and studied in [2] and [4]. It is useful in the study of Hamiltonian graphs. The definition of $sc(G)$ is similar to that of $c(G)$.

**LEMMA 2.2.** $sc(G)$ is well-defined.

**PROOF.** This follows from the proof of ([2], p. 56, Lemma 4.4.2).

**THEOREM 2.3.** A graph is Hamiltonian-connected if and only if its strong closure is Hamiltonian-connected.

**PROOF.** This follows immediately from Theorem 2.1 and Lemma 2.2. Theorem 2.3 gives some interesting results.

**COROLLARY 2.4.** If $sc(G)$ is a complete graph, then $G$ is Hamiltonian-connected.

**PROOF.** If $sc(G)$ is complete, then it is Hamiltonian-connected and so by Theorem 2.3, $G$ is also Hamiltonian-connected.

The following result was obtained by O. Ore (see [1], p. 136, Theorem 11.3 or [5]).

**COROLLARY 2.5.** If $d(u) + d(v) \geq n+1$ for every pair of nonadjacent vertices $u$ and $v$, then $G$ is Hamiltonian-connected.

**PROOF.** Since $d(u) + d(v) \geq n+1$ for every pair of nonadjacent vertices $u$ and $v$, it follows that $Sc(G)$ is a complete graph. Therefore by Corollary 2.4, $G$ is Hamiltonian-connected.

If $G$ has vertices $v_1, v_2, \ldots , v_n$, the sequence $(d(v_1), d(v_2), \ldots , d(v_n))$ is called a degree sequence of $G$. The following result is similar to a result obtained by Chvátal (see [2], p 57, Theorem 4.5).
COROLLARY 2.6. Let \((d_1, d_2, \ldots, d_n)\) be a degree sequence of \(G\) such that \(d_1 \leq d_2 \leq \cdots \leq d_n\). Suppose that there is no value of \(p\) less than \(\frac{1}{2} (n + 1)\) for which \(d_p \geq p\) and \(d_{n-p} < n - (p-1)\). Then \(G\) is Hamiltonian-connected.

PROOF. By a similar argument as in the proof of ([2], p. 57, Theorem 4.5), we can show that \(sc(G)\) is a complete graph. Therefore by Corollary 2.4, \(G\) is Hamiltonian-connected.

3. AN ALGORITHM FOR FINDING STRONG CLOSURE.

In this section, we give an algorithm to find \(sc(G)\). Let \(V(G) = \{u_1, u_2, \ldots, u_n\}\).

STEP 1. For \(1 \leq i < j \leq n\), let

\[
f(i, j) = \begin{cases} 
    d(u_i) + d(v_j), & \text{if } u_i \notin D(v_j) \\
    0, & \text{if } u_i \in D(v_j)
  \end{cases}
\]

STEP 2. Choose \(f(I, J) = \max \{f(i, j) : 1 \leq i < j \leq n\}\).

If \(f(I, J) < n + 1\), then go to Step 4.

STEP 3. \(f(I, J) = 0\).

If \(f(p, I) \neq 0\), then \(f(p, I) = f(p, I) + 1 (1 \leq p < I)\).

If \(f(I, p) \neq 0\), then \(f(I, p) = f(I, p) + 1 (1 \leq p < n)\).

If \(f(q, J) \neq 0\), then \(f(q, J) = f(q, J) + 1 (1 \leq q < J)\).

If \(f(J, q) \neq 0\), then \(f(J, q) = f(J, q) + 1 (J < q \leq n)\).

Go to Step 2.

STEP 4. Form \(sc(G)\) by joining \(u_i\) to \(u_j\) if \(f(i, j) = 0 (1 \leq i < j \leq n)\).

Let \(G\) be represented by an adjacency matrix. Steps 1 and 4 can be implemented in \(O(n^2)\) time. Clearly, Step 3 runs in \(O(n)\) time. Let \(K = \{(i, j) : f(i, j) \neq 0 \quad 1 \leq i < j \leq n\}\). Then

\[
|K| = 1 + 2 + \cdots + (n-1) - m = \frac{1}{2} (n^2 - n - 2m).
\]

By using F-heaps data structure [3], find \(\max \{f(i, j)\}\) takes \(O(\log_2 |K|) = O(\log_2 n)\) time. Hence Steps 2 and 3 take \(O(|K| (n + \log_2 n)) = O(n |K|)\). Thus overall we have an \(O(n |K|)\) algorithm.

LEMMA 3.1. If \(n \geq 4\) and \(d(u) \leq 2\) in \(G\), then \(d(u) \leq 2\) in \(sc(G)\).

PROOF. Let \(v\) be a vertex of \(G\) which is not adjacent to \(u\). Then \(d(v) \leq n - 2\). Hence \(d(u) + d(v) \leq 2 + (n - 2) = n\) and so Lemma 3.1 is true.

Lemma 3.1 allows us not to consider \(u\) in the computation of \(sc(G)\) if \(d(u) \leq 2\) in \(G\).

REFERENCES
