SUBRINGS OF I-RINGS AND S-RINGS

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ABSTRACT. Let $R$ be a non-commutative associative ring with unity $1 \neq 0$, a left $R$-module is said to satisfy property (I) (resp. (S)) if every injective (resp. surjective) endomorphism of $M$ is an automorphism of $M$. It is well known that every Artinian (resp. Noetherian) module satisfies property (I) (resp. (S)) and that the converse is not true. A ring $R$ is called a left I-ring (resp. S-ring) if every left $R$-module with property (I) (resp (S)) is Artinian (resp. Noetherian). It is known that a subring $B$ of a left I-ring (resp. S-ring) $R$ is not in general a left I-ring (resp. S-ring) even if $R$ is a finitely generated $B$-module, for example the ring $M_3(K)$ of $3 \times 3$ matrices over a field $K$ is a left I-ring (resp S-ring), whereas its subring

$$B = \left\{ \begin{bmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & 0 & \alpha \end{bmatrix} / \alpha, \beta, \gamma \in K \right\}$$

which is a commutative ring with a non-principal Jacobson radical

$$J = K. \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + K. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is not an I-ring (resp. S-ring) (see [4], theorem 8). We recall that commutative I-rings (resp S-rings) are characterized as those whose modules are a direct sum of cyclic modules, these rings are exactly commutative, Artinian, principal ideal rings (see [1]). Some classes of non-commutative I-rings and S-rings have been studied in [2] and [3]. A ring $R$ is of finite representation type if it is left and right Artinian and has (up to isomorphism) only a finite number of finitely generated indecomposable left modules. In the case of commutative rings or finite-dimensional algebras over an algebraically closed field, the classes of left I-rings, left S-rings and rings of finite representation type are identical (see [1] and [4]). A ring $R$ is said to be a ring with polynomial identity (P. I-ring) if there exists a polynomial $f(X_1, X_2, \ldots, X_n)$, $n \geq 2$, in the non-commuting indeterminates $X_1, X_2, \ldots, X_n$ over the center $Z$ of $R$ such that one of the monomials of $f$ of highest total degree has coefficient 1, and $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n$ in $R$. Throughout this paper all rings considered are associative rings with unity, and by a module $M$ over a ring $R$ we always understand a unitary left $R$-module. We use $M_R$ to emphasize that $M$ is a unitary right $R$-module.

KEY WORDS AND PHRASES: Left I-ring, left S-ring, ring with polynomial identity, ring of finite representation type.

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1. THE MAIN RESULT

THEOREM. Let \( R \) be a left I-ring (resp. S-ring), and \( B \) be a sub-ring of \( R \) contained in the center \( Z \) of \( R \). Suppose that \( R \) is a finitely generated flat \( B \)-module. Then \( B \) is an I-ring (resp. S-ring).

To prove this theorem we need some results.

It is easy to see that

**LEMMA 1.** Every homomorphic image of a left I-ring (resp. S-ring) is a left I-ring (resp. S-ring).

**LEMMA 2.** Let \( P_1 \) and \( P_2 \) be two prime ideals of a ring \( R \). If \( P_1 \) is not contained in \( P_2 \) then \( \text{Hom}_R(R/P_1, R/P_2) \neq \{0\} \).

**PROOF.** Let \( f : R/P_1 \to R/P_2 \) be an \( R \)-homomorphism, and set \( f(1 + P_1) = t + P_2 \), where \( t \in R \). Let \( x \in P_1 \setminus P_2 \), and let \( r \) be any element in \( R \). We have \( P_2 = f(xr + P_1) = xrt + P_2 \). Thus \( xrt \in P_2 \). Since \( P_2 \) is prime, we have \( t \notin P_2 \), and hence \( f = 0 \).

**LEMMA 3.** Let \( R \) be a prime ring with polynomial identity. If \( R \) is a left I-ring (resp. S-ring), then \( R \) is simple Artinian.

**PROOF.** Let \( R' \) be the total ring of fractions of \( R \) \([5]\). It is known that \( R' \) is simple Artinian \([5]\), so the \( R \)-module \( R' \) satisfies (I) (resp. (S)). Since \( R \) is a left I-ring (resp. S-ring), then \( R' \) is an Artinian (resp. Noetherian) \( R \)-module and hence \( R' = R \).

**LEMMA 4.** Let \( R \) be a semi-prime ring with polynomial identity. If \( R \) is a left I-ring (resp. S-ring), then \( R \) is semi-simple Artinian.

**PROOF.** Let \( (P_t)_{t \in L} \) be a family pairwise distinct minimal prime ideals of \( R \) such that \[ t \in L \]

Then it follows from Lemma 3 that the rings \( R/P_t(\ell \in L) \) are simple Artinian, so the left \( R \)-modules \( R/P_t(\ell \in L) \) satisfy (I) (resp. (S)). Following Lemma 1, \( \text{Hom}_R(R/P_t, R/P_{t'}) = \{0\} \) for \( \ell \neq t' \), so the left \( R \)-module \( M = \oplus_{t \in L} R/P_t \) satisfies (I) (resp. (S)). Since \( R \) is a left I-ring (resp. S-ring), then \( M \) is Artinian. But \( R \) regarded as left \( R \)-module is isomorphic to a submodule of the semi-simple Artinian left \( R \)-module \( M \), hence \( R \) is semi-simple Artinian.

**PROPOSITION 5.** Let \( R \) be a ring with polynomial identity. If \( R \) is a left S-ring (resp. I-ring), then \( R \) is left Artinian.

**PROOF.** Suppose that \( R \) is a left S-ring (resp. I-ring) then the quotient ring \( R/\text{rad}(R) \), where \( \text{rad}(R) \) is the prime radical of \( R \), is a left S-ring (resp. I-ring), so, following Lemma 4, the ring \( R/\text{rad}(R) \) is semi-simple Artinian. This fact implies that \( R \) is semi-perfect and hence \( \text{rad}(R) = J(R) \), where \( J(R) \) is the Jacobson radical of \( R \). Let \( e \) be a primitive idempotent of \( R \). Since the endomorphism ring of the \( R \)-module \( Re \) is isomorphic to the local ring \( eRe \) with a nil Jacobson radical \( eJ(R)e \), then the \( R \)-module \( Re \) satisfies property (I) (resp. (S)). It follows that the \( R \)-module \( Re \) is Noetherian (resp. Artinian). Since \( R \) regarded as \( R \)-module is a direct sum of finitely many left \( R \)-modules of the form \( Re \), where \( e \) is a primitive idempotent of \( R \), then \( R \) is Noetherian. Let \( P \) now be a prime ideal of \( R \). Since the prime ring \( R/P \) is simple in virtue of Lemma 3, then \( R \) is left Artinian.

**PROOF OF THE MAIN THEOREM.** Since \( R \) is a finitely generated \( Z \)-module, then \( R \) is a ring with polynomial identity (see \([6]\)). So by Proposition 5 \( R \) is a left Artinian ring. Thus by \([7]\) the ring \( B \) is Artinian. Let \( e_1, \ldots, e_n \) be primitive idempotents of \( B \) such that \( B = \oplus_{i=1}^n e_iBe_i \). For every \( i, 1 \leq i \leq n \), \( B_i = e_iBe_i \) is a local Artinian ring. To show that \( B \) is a left I-ring (resp. S-ring) it is enough to show that for every \( i, 1 \leq i \leq n \), \( B_i \) is a left I-ring (resp. S-ring). We have \( A = \oplus_{i=1}^n A_i \), where \( A_i = e_iBe_i, 1 \leq i \leq n \). By hypothesis the left \( B \)-module \( \oplus_{i=1}^n A_i = A \) is flat and finitely generated, so the \( B \)-module \[ A_i = e_iBe_i \cong e_iBe_i \oplus B = A \oplus_{i=1}^n e_iBe_i = A \oplus_{i=1}^n B_i, \]
is also flat and finitely generated. Since $B_i$ is an Artinian local ring then the $B_i$-module $A_i$ is faithfully flat (see [8] proposition 1, p. 44).

Suppose now that $B_i$ is not an I-ring (resp. S-ring) for some $i, 1 \leq i \leq n$. Then by Proposition 2 of [2], there exists a $B_i$-module $M$ of infinite length such that, for every integer $n \geq 1$, the $B_i$-module $M^n$ satisfies both properties (I) and (S). Following [8] (corollary 2, p. 107), the $B_i$-module $A_i$ is a free module. Let $M' = M \otimes_{B_i} A_i$. Since the $B_i$-module $M$ is of infinite length and $A_i$ is a faithfully flat $B_i$-module, then $M'$ is an $A_i$-module of infinite length. On the other hand, since $A_i$ is a free $B_i$-module, there exists an integer $s \geq 1$ such that $A_i = B_i^s$. We have then the $B_i$-module isomorphism

$$M' = M \otimes_{B_i} A_i = M \otimes_{B_i} B_i^s \cong M^s.$$ 

Hence the $B_i$-module $M' \cong M^s$ satisfies both properties (I) and (S) and therefore $M'$, regarded as $A_i$-module, satisfies properties (I) and (S). This fact implies that the homomorphic image $A_i$ of the left I-ring (resp. S-ring) $A$ is not a left I-ring (resp. S-ring), in contradiction with Lemma 1.

**COROLLARY.** Let $R$ be a left I-ring (resp. S-ring). If $R$ is a finitely generated flat module over its center $Z$, then $Z$ is an I-ring (resp. S-ring).

The following example shows that the converse of the theorem above is not true. Let $K$ be a field. The commutative ring $A = K[X,Y]/(X^2, XY, Y^2)$ is not an I-ring (resp. S-ring) because its Jacobson radical $J = KX + KY$ is not principal (see [1], theorem 8). On the other hand $K$ is an I-ring (resp. S-ring) and $A$ is a finite-dimensional $K$-vector space.

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**REFERENCES**


