ORDERED COMPACTIFICATIONS AND FAMILIES OF MAPS

D. M. LIU and D. C. KENT
Department of Pure and Applied Mathematics
Washington State University
Pullman, WA 99163-3113, U.S.A.

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ABSTRACT. For a $T_{3.5}$-ordered space, certain families of maps are designated as “defining families.” For each such defining family we construct the smallest $T_2$-ordered compactification such that each member of the family can be extended to the compactification space. Each defining family also generates a quasi-uniformity on the space whose bicompletion produces the same $T_2$-ordered compactification.

KEY WORDS AND PHRASES. $T_{3.5}$-ordered space, $T_2$-ordered compactification, defining family of maps, quasi-uniform space, bicompletion.

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INTRODUCTION.

Let $X$ be a $T_{3.5}$-ordered space, and let $CI^*(X)$ be the set of all increasing, continuous maps from $X$ into $[0, 1]$. A subset $\Phi$ of $CI^*(X)$ which induces both the weak order and weak topology on $X$ is called a defining family for $X$. For each such defining family $\Phi$, we construct the smallest $T_2$-ordered compactification $K_\Phi$ with the property that each member of $\Phi$ can be extended to $K_\Phi$. If $\Phi_1$ and $\Phi_2$ are two defining families for $X$ such that $\Phi_1 \subseteq \Phi_2$, then $K_{\Phi_1} \leq K_{\Phi_2}$. For each defining family $\Phi$, there is a largest defining family $\hat{\Phi}$ such that $K_\Phi = K_{\hat{\Phi}}$. Those defining families which are $\hat{\Phi}$ for some defining family $\Phi$ are called maximal defining families, and if $\Phi$ and $\Psi$ are two maximal defining families, $K_\Phi \leq K_\Psi$ iff $\Phi = \Psi$. The largest defining family for $X$ is $CI^*(X)$, and if $\Phi = CI^*(X)$ then $K_\Phi$ is the Nachbin (or Stone-Čech ordered) compactification [2].

Each defining family $\Phi$ also generates a quasi-uniformity $\nu_\phi$ on $X$ (related to the “usual” quasi-uniformity $\nu_0$ on $[0, 1]$) which is $T_0$ and totally bounded. The bicompletion of $(X, \nu_\phi)$ (as defined in [1]) yields a uniform ordered space which, in turn, gives the compactification $K_\phi$. The maximal defining family $\hat{\Phi}$ is precisely the set of all quasi-uniformly continuous maps from $(X, \nu_\phi)$ into $([0, 1], \nu_0)$.

1. PRELIMINARIES.

If $X$ is a set, we denote by $F(X)$ the set of all (proper) filters on $X$ and by $UF(X)$ the set of all ultrafilters on $X$. A non-empty collection $\mathcal{G}$ of subsets of $X$ is called a grill on $X$ if: (1) $\emptyset \notin \mathcal{G}$; (2) $A \in \mathcal{G}$ and $A \subseteq B$ implies $B \in \mathcal{G}$; (3) $A \cup B \in \mathcal{G}$ implies $A \in \mathcal{G}$ or $B \in \mathcal{G}$. With every $\mathcal{F} \in F(X)$, we associate the grill $\gamma(\mathcal{F}) = \{A \subseteq X : X \setminus A \notin \mathcal{F}\}$; equivalently, $\gamma(\mathcal{F})$ is the union of all ultrafilters finer than $\mathcal{F}$.

Let $(X, \leq)$ be a poset; a subset $A \subseteq X$ is increasing (respectively, decreasing) if $x \in A$ and $x \leq y$ (respectively, $y \leq x$) implies $y \in A$. If $(X, \leq)$ and $(Y, \leq^*)$ are posets, then a mapping $f :
(X, ≤) → (Y, ≤*) is increasing (respectively, decreasing) if x ≤ y implies f(x) ≤* f(y) (respectively, f(y) ≤* f(x)).

An ordered space (X, τ, ≤) consists of a poset (X, ≤) and a topology τ on (X, ≤) which is convex (meaning that the collection of all τ-open sets which are either increasing or decreasing is a subbase for τ). Usually an ordered space (X, τ, ≤) will simply be denoted by X. The closed unit interval [0, 1] with its usual order and topology is designated by I. For any ordered space X, let CI*(X) (respectively, CD*(X)) be the set of all continuous increasing (respectively, decreasing) maps from X into I. More generally, for ordered spaces X and Y, CI(X, Y) represents the set of all continuous, increasing functions from X into Y.

An ordered space X is said to be T2-ordered if the order "≤" is closed in X × X. A T2-ordered space X which has both the weak order (see Condition W0 below) and weak topology induced by CI*(X) is said to be T3.s-ordered (or completely regular ordered in the terminology of [2]). Some well-known characterizations of T3.s-ordered spaces are summarized in the following proposition.

**PROPOSITION 1.1** The following statements about an ordered space X are equivalent.

1. X is T3.s-ordered.
2. X is a subspace of a compact, T2-ordered space.
3. X satisfies the following conditions:
   (i) If x ∈ X, A is a closed subset of X, and x ∈ A, then there is f ∈ CI*(X) and g ∈ CD*(X) such that f(x) = g(x) = 0 and f(y) ∨ g(y) = 1, for all y ∈ A;
   (ii) If x ∈ y in X, there is f ∈ CI*(X) such that f(y) = 0 and f(x) = 1.
4. The order and topology for X are induced by some quasi-uniformity W on X (i.e., ∩W is the order for X and the topology of X is the uniform topology of the uniformity W ∪ W−1).

Every T3.s-ordered space X has a largest T2-ordered compactification βX called the Nachbin compactification, which can be constructed by embedding X in the "ordered cube" ICI*(X), with the product order and topology.

Let X be an ordered space. If F is any subset of CI*(X) such that X has the weak order and the weak topology determined by F, then F is called a defining family for X. More precisely, F ⊆ CI*(X) is a defining family if the following conditions are satisfied:

(We) For any F ∈ UF(X), F → x in X iff f(F) → f(x) in I, for all f ∈ F.
(We) For any (x, y) ∈ X × X, x ≤ y in X iff f(x) ≤ f(y) in I, for all f ∈ F.

Some rather obvious remarks about defining families are summarized in the next proposition.

**PROPOSITION 1.2** Let X be an ordered space.

1. X is T3.s-ordered iff X allows at least one defining family. In particular, CI*(X) is a defining family for every T3.s-ordered space.
2. If F1 ⊆ F2 ⊆ CI*(X) and F1 is a defining family for X, then F2 is also a defining family for X.

2. THE COMPACTIFICATION KΦ.

Let X be a T3.s-ordered space. If F ∈ UF(X) and f ∈ CI*(X), there is a unique point aF,f in I such that f(F) → aF,f. For any a ∈ I, let V(a) denote the neighborhood filter at a. If F is a defining family for X and F ∈ UF(X), we define the filter FΦ = ∨{f−1(V(aF,f)) : f ∈ F}. Note that if F → x in X, then aF,f = f(x) for all f ∈ F, and in this case FΦ is simply the neighborhood filter at x.

Continuing with the assumptions of the preceding paragraph, let XΦ = {γ(FΦ) : F ∈ UF(X)} be the set of grills associated with the filters FΦ. If γ ∈ XΦ and F, G ∈ UF(X) are such that F ⊆ γ and G ⊆ γ, then aF,f = aG,f, for all f ∈ F. It therefore follows that, for each f ∈ F, the function
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\( f_\Phi : X_\Phi \to I, \) defined by \( f_\Phi(\gamma) = a_{\mathcal{F}_\Phi}, \) where \( \mathcal{F} \) any ultrafilter that is a subset of \( \gamma, \) is well-defined. If \( i_\Phi : X \to X_\Phi \) is defined by \( i_\Phi(x) = \gamma(x_\Phi), \) where \( x_\Phi, \) is the fixed ultrafilter generated by \{\{x\}\}, then clearly \( i_\Phi \) is an injection and the diagram below commutes for every \( f \in \Phi. \)

\[
\begin{array}{c}
X_\Phi \\
\downarrow \phi \\
I
\end{array}
\quad
\begin{array}{c}
X \\
\downarrow f
\end{array}
\]

Let \( X_\Phi \) be equipped with the weak order and weak topology induced by \( \{f_\Phi : f \in \Phi\}. \) Then \( i_\Phi \) is an ordered space embedding (i.e., \( i_\Phi \) is topological embedding, and \( x \leq y \Leftrightarrow i_\Phi(x) \leq i_\Phi(y), \) where \( \leq_\Phi \) denotes the order of \( X_\Phi). \)

**THEOREM 2.1** Let \( X \) be a \( T_{3.5}\)-ordered space and \( \Phi \) a defining family for \( X. \) Then \( (X_\Phi, i_\Phi) \) is a \( T_2\)-ordered compactification of \( X, \) and each \( f \in \Phi \) has a unique, continuous, increasing extension to \( X_\Phi \) such that the diagram below commutes.

\[
\begin{array}{c}
X_\Phi \\
\downarrow \phi \\
I
\end{array}
\quad
\begin{array}{c}
X \\
\downarrow f
\end{array}
\]

**PROOF.** The family \( \Phi^\gamma = \{f_\Phi : f \in \Phi\} \) separates points in \( X_\Phi, \) and therefore \( X_\Phi \) is \( T_{3.5}\)-ordered; in particular, \( X_\Phi \) is \( T_2\)-ordered. In view of the paragraph preceding the theorem, it remains only to show that \( X_\Phi \) is compact and \( i_\Phi(X) \) is dense in \( X_\Phi. \)

Let \( A \in \text{UF}(X). \) For each \( \gamma \in X_\Phi, \) choose an ultrafilter \( \mathcal{F}, \) such that \( \mathcal{F} \subseteq \gamma; \) in particular, if \( \gamma = \gamma(\mathcal{F}_\Phi) \) where \( \mathcal{F} \to x \) in \( X, \) define \( \mathcal{F}_\gamma = \hat{x}. \) If \( B \subseteq X, \) let \( B^* = \{\gamma \in X_\Phi : B \in \mathcal{F}_\gamma\}. \) Then, define \( \mathcal{F}_A = \{A \subseteq X : A^* \subseteq A\}; \) one easily verifies that \( \mathcal{F}_A \) is an ultrafilter. We shall show that \( A \to \gamma(\mathcal{F}_A) \) in \( X_\Phi. \) For this purpose, it suffices to show that \( f_\Phi(A) \to f_\Phi(\gamma(\mathcal{F}_A)) = a_{\mathcal{F}_A,f}. \) for all \( f \in \Phi. \) Given \( f \in \Phi, \) let \( U \) be a closed neighborhood of \( a_{\mathcal{F}_A,f} \) in \( I. \) We first observe that \( f(\mathcal{F}_A) \to a_{\mathcal{F}_A,f}, \) and hence \( f^{-1}(U) \in \mathcal{F}_A, \) which implies \( (f^{-1}(U))^* \subseteq A. \) Then note that \( f(\mathcal{F}_A) \to a_{\mathcal{F}_A,f}; \) consequently \( f^{-1}(U) \subseteq A, \) and \( f_\Phi(A) \to a_{\mathcal{F}_A,f}. \) Thus \( X_\Phi \) is compact.

Finally, let \( \gamma \in X_\Phi \) and, for \( B \subseteq X, \) let \( B^* \) be defined as in the preceding paragraph. If \( \mathcal{F} \in \text{UF}(X) \) and \( \mathcal{F} \subseteq \gamma, \) let \( \mathcal{F}^* \) be the filter on \( X_\Phi \) generated by \( \{F^* : F \in \mathcal{F}\}. \) One easily shows that \( \mathcal{F}^* \to \gamma \) in \( X_\Phi. \) Since \( i_\Phi(F) \geq \mathcal{F}^*, \) it follows that \( i_\Phi(X) \) is dense in \( X_\Phi. \)

The compactification \( (X_\Phi, i_\Phi) \) of \( X \) determined by a defining family \( \Phi \) will be denoted by \( K_\Phi. \)

By the preceding theorem, each \( f \in \Phi \) has a unique extension \( f_\Phi \in CI^*(X_\Phi). \) If \( Y \) is any compact, \( T_2\)-ordered space, we define \( CI_\Phi(X,Y) = \{f \in CI(X,Y) : h \circ f \in \Phi, \text{ for all } h \in CI^*(Y)\}. \) The next theorem establishes that each \( f \in CI_\Phi(X,Y) \) can be “lifted” relative to \( K_\Phi. \)

**THEOREM 2.2** Let \( X \) be a \( T_{3.5}\)-ordered space, \( \Phi \) a defining family for \( X, \) and \( Y \) a compact, \( T_2\)-ordered space. If \( g \in CI_\Phi(X,Y), \) then there is a unique \( g_\Phi \in CI(X_\Phi,Y) \) such that the diagram below commutes.

\[
\begin{array}{c}
X_\Phi \\
\downarrow \phi \\
Y
\end{array}
\quad
\begin{array}{c}
X \\
\downarrow g
\end{array}
\]

**PROOF.** Let \( g \in CI_\Phi(X,Y), \) and \( \gamma \in X_\Phi; \) assume \( \mathcal{F} \) is an ultrafilter and \( \mathcal{F} \subseteq \gamma. \) Define \( g_\Phi : X_\Phi \to Y \) as following: \( g_\Phi(\gamma) = y_{x,g}, \) where \( y_{x,g} \) is the unique limit of \( g(\mathcal{F}) \) in \( Y. \) Using the facts that \( g \in CI_\Phi(X,Y) \) and \( CI^*(Y) \) separates points in \( Y, \) we see that \( g_\Phi(\gamma) \) is independent of the ultrafilter \( \mathcal{F} \) which represents \( \gamma, \) so \( g_\Phi \) is well defined.
If \( h \in CI'(Y) \), let \( h' = h \circ g \). Then we observe that the preceding definition of \( g_* \) makes the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow{g} & \nearrow{h'} & \\
X & \xrightarrow{g_*} & Y
\end{array}
\]

If \( \gamma \leq \delta \) in \( X_\Phi \), then \( h'_\Phi(\gamma) \leq h'_\Phi(\delta), \forall h \in CI'(Y) \), which implies \( h(\gamma) \leq h(\delta) \) holds for all \( h \in CI'(Y) \). Since \( Y \) has the weak order induced by \( CI'(Y) \), \( g_*(\gamma) \leq g_*(\delta) \). Thus \( g_* \) is increasing. A similar argument, based on \( Y \) having the weak topology induced by \( CI'(Y) \), shows that \( g_* \) is continuous. The uniqueness of \( g_* \) is obvious because all spaces involved are Hausdorff. \( \blacksquare \)

We omit the simple proof of the next proposition.

**PROPOSITION 2.3** If \( \Phi \) is a defining family for a \( T_{3.5} \)-ordered space \( X \), then \( \Phi' = \{ f_\Phi : f \in CI'(X) \} \) is a defining family for \( X_\Phi \).

Starting with a \( T_{3.5} \)-ordered space \( X \) and a defining family for \( X \), it follows that \( \Phi' \) and \( CI'(X_\Phi) \) are both defining families for \( X_\Phi \), and it is clear that \( \Phi' \subseteq CI'(X_\Phi) \). Let \( \hat{\Phi} = \{ f \in CI'(X) : \text{there is } g \in CI'(X_\Phi) \text{ such that } f = g \circ i_\Phi \} \); in other words, \( \hat{\Phi} \) consists of all members of \( CI'(X) \) which have a continuous, increasing extension in \( CI'(X_\Phi) \). Clearly \( \hat{\Phi} \subseteq \hat{\Phi} \), and so \( \hat{\Phi} \) is a defining family for \( X \). Note that \( (\hat{\Phi})' = CI'(X_\Phi) \), and since \( (\hat{\Phi})' \) is, by Proposition 2.3, a defining family for \( X_\Phi \), it follows that \( X_\Phi = X_\Phi \). These observations yield the following result.

**PROPOSITION 2.4** If \( \Phi \) is a defining family for a \( T_{3.5} \)-ordered space \( X \), then \( \hat{\Phi} = \{ f \in CI'(X) : \text{there is } g \in CI'(X_\Phi) \text{ such that } f = g \circ i_\Phi \} \) is the largest defining family for \( X \) such that \( K_\Phi = K_{\hat{\Phi}} \).

**THEOREM 2.5** Let \( \Phi, \Psi \) be defining families for a \( T_{3.5} \)-ordered space.

(a) If \( \Phi \subseteq \Psi \), then \( K_\Phi \subseteq K_\Psi \).

(b) \( K_\Phi \leq K_\Psi \) iff \( \hat{\Phi} \subseteq \hat{\Psi} \).

**PROOF.** (a) \( \Phi \subseteq \Psi \) implies \( \hat{\Phi} \subseteq \hat{\Psi} \). Considering the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i_\Phi} & X_\Phi \\
\downarrow{(i_\Phi)_\Psi} & & \downarrow{(i_\Phi)_\Phi} \\
X & \xrightarrow{i_\Phi} & X_\Phi
\end{array}
\]

and applying Theorem 2.2, we see that \( (i_\Phi)_\Psi \) is increasing and continuous. Thus \( K_\Phi = K_{(i_\Phi)_\Phi} \leq K_{(i_\Phi)_\Psi} = K_\Psi \).

(b) If \( K_\Phi \leq K_\Psi \), then there is an increasing, continuous map \( \sigma \) making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i_\Phi} & X_\Psi \\
\downarrow{i_\Phi} & \nearrow{\sigma} & \\
X & \xrightarrow{i_\Phi} & X_\Phi
\end{array}
\]

commute. Each member of \( \hat{\Phi} \) has the form \( f \circ i_\Phi \) for some \( f \in CI'(X_\Phi) \). But \( f \circ i_\Phi = f \circ \sigma \circ i_\Phi \) is also in \( \hat{\Psi} \). Thus \( \hat{\Phi} \subseteq \hat{\Psi} \). The converse follows from (a). \( \blacksquare \)

If \( X \) is a \( T_{3.5} \)-ordered space, let \( DF(X) \) be the poset of all defining families, ordered by inclusion. Two defining families \( \Phi \) and \( \Psi \) in \( DF(X) \) are equivalent if \( K_\Phi = K_\Psi \) (i.e., if \( K_\Phi \) and \( K_\Psi \) are equivalent compactifications of \( X \) in the usual sense). Thus \( DF(X) \) is partitioned into equivalent
classes, and each equivalent class \((\Phi)\) contains a largest member \(\hat{\Phi}\) which we call a maximal defining family.

**COROLLARY 2.6** Let \(X\) be a \(T_{3.5}\)-ordered space, \(K = (X', \phi)\) a \(T_2\)-ordered compactification of \(X\), and \(\Phi \in DF(X)\) such that each \(f \in \Phi\) has an extension \(f' \in Cl'(X')\). Then \(K_\Phi \leq K\).

**COROLLARY 2.7** For a \(T_{3.5}\)-ordered space, the correspondence \(\Phi \mapsto K_\Phi\) is bijective and order-preserving between the maximal defining families for \(X\) and the \(T_2\)-ordered compactifications of \(X\).

3. DEFINING FAMILIES AND QUASI-UNIFORMITIES.

This concluding section is based on the results of Fletcher and Lindgren [1], and to some extent we borrow their notation.

Let \((X, \mathcal{V})\) be a quasi-uniform space; the associated uniformity \(\mathcal{V} \vee \mathcal{V}^{-1}\) will be denoted by \(\mathcal{V}'\).

Recall that \((X, \mathcal{V})\) is \(T_0\) iff \(\mathcal{V}\) is a partial order (or, equivalently, \((X, \mathcal{V}')\) is \(T_2\)), and totally bounded iff, for each \(U \in \mathcal{V}\), there is a finite cover \(\{A_1, \cdots, A_n\}\) of \(X\) such that \(A_i \times A_j \subseteq U\), for \(i = 1, \cdots, n\). Note that \(\mathcal{V}\) is totally bounded iff \(\mathcal{V}'\) is totally bounded.

Every \(T_0\), quasi-uniform space \((X, \mathcal{V})\) induces a uniform ordered space \((X, \mathcal{U}, \leq)\), where \(\mathcal{U} = \mathcal{V}'\) and \(\leq = \cap \mathcal{V}\); also associated with \((X, \mathcal{V})\) is the \(T_{3.5}\)-ordered space \((X, \tau, \leq)\), where \(\tau = eq\) and \(\leq\) is again \(\cap \mathcal{V}\). Furthermore, for every compact, \(T_2\)-ordered space \((X, \tau, \leq)\), there is a unique quasi-uniformity \(\mathcal{V}\) on \(X\) such that \(\tau = eq\) and \(\leq = \cap \mathcal{V}\) (Theorem 4.21, [1]). In particular, for the compact, \(T_2\)-ordered space \(I\), the unique compatible quasi-uniformity, denoted here by \(\mathcal{W}\), has a base of sets of the form \(W_\epsilon = \{(x, y) \in I \times I : |x - y| \leq \epsilon\}\), where \(\epsilon > 0\).

For a quasi-uniform space \((X, \mathcal{V})\), let \(QUC(X, \mathcal{V})\) be the set of all quasi-uniformly continuous maps from \((X, \mathcal{V})\) into \((I, \mathcal{W})\). If \(X = (X, \tau, \leq)\) is the \(T_{3.5}\)-ordered space associated with \((X, \mathcal{V})\), it is clear that \(QUC(X, \mathcal{V}) \subseteq Cl'(X)\). It is shown in Theorems 3.29 and 3.33 of [1] that every \(T_0\), quasi-uniform space \((X, \mathcal{V})\) has a bicompletion \(((X, \mathcal{V}), j)\) such that \(((\hat{X}, \hat{\mathcal{V}}), j)\) is the unique uniform space completion of \((X, \mathcal{V}')\), and each \(f \in QUC(X, \mathcal{V})\) has a unique extension in \(QUC(\hat{X}, \hat{\mathcal{V}})\). These observations lead to the following proposition.

**PROPOSITION 3.1** Let \((X, \mathcal{V})\) be a \(T_0\), totally bounded quasi-uniform space with associated \(T_{3.5}\)-ordered space \((X, \tau, \leq)\), and let \(((\hat{X}, \hat{\mathcal{V}}), j)\) be the bicompletion of \((X, \mathcal{V})\). If \(((\hat{X}, \hat{\mathcal{V}}), j)\) is the \(T_{3.5}\)-ordered space associated with \((X, \mathcal{V})\), then \(\hat{K} = ((\hat{X}, \hat{\mathcal{V}}), j)\) is a \(T_2\)-ordered compactification of \((X, \tau, \leq)\).

**THEOREM 3.2** Let \(X\) be a \(T_{3.5}\)-ordered space and \(\Phi \in DF(X)\). Let \(\mathcal{V}_\Phi\) be the weak uniformity on \(X\) induced by \(\Phi\) relative to \((I, \mathcal{W})\). Let \(((\hat{X}_\Phi, \hat{\mathcal{V}}_\Phi), j)\) be the bicompletion of \((X, \mathcal{V}_\Phi)\), and \(\hat{K}_\Phi = ((\hat{X}_\Phi, \hat{\mathcal{V}}_\Phi), j)\) be the \(T_2\)-ordered compactification of \(X\) induced by the bicompletion. Then \(\hat{K}_\Phi = K_\Phi\).

**PROOF.** Let \(\mathcal{V}\) be the unique, \(T_0\) totally bounded quasi-uniformity on \(X_\Phi\), whose associated \(T_{3.5}\)-ordered space is the compactification \(((X_\Phi, \tau_\Phi, \leq_\Phi), i_\Phi)\) derived from \(\Phi\). The latter space has the weak order and topology induced by \(\Phi'\) (see Proposition 2.3) relative to \(I\), and hence \(\mathcal{V}\) is the weak quasi-uniformity on \(X_\Phi\) induced by \(\Phi'\) relative to \((I, \mathcal{W})\). If \(U = (i_\Phi)^{-1}(\mathcal{V})\) is the restriction of \(\mathcal{V}\) to \(X\), then \(U\) is the weak quasi-uniformity on \(X\) induced by \(\Phi\) relative to \((I, \mathcal{W})\). In other words, \(U = \mathcal{V}_\Phi\). Since the \(T_2\)-ordered compactification associated with a \(T_0\), totally bounded quasi-uniformity is unique (up to equivalence), \(\hat{K}_\Phi = K_\Phi\).

**COROLLARY 3.3** Let \(X\) be a \(T_{3.5}\)-ordered space and \(\Phi \in DF(X)\). Then \(\hat{\Phi} = QUC(X, \mathcal{V}_\Phi)\).

**PROOF.** By Theorem 3.29, [1], each \(f \in QUC(X, \mathcal{V}_\Phi)\) can be extended to the compactification \(\hat{K}_\Phi = K_\Phi\); thus \(QUC(X, \mathcal{V}_\Phi) \subseteq \hat{\Phi}\). Conversely, each \(f \in \hat{\Phi}\) has a unique, increasing, continuous extension to \(K_\Phi = \hat{K}_\Phi\), and this extension of \(f\) is quasi-uniformly continuous from \((\hat{X}_\Phi, \hat{\mathcal{V}}_\Phi)\) into
(I, W). Thus $f \in QUC(X, \mathcal{V}_\Phi)$. \hfill \Box

**COROLLARY 3.4** Let $(X, \mathcal{V})$ be a $T_0$, totally bounded quasi-uniform space with associated compact, $T_2$-ordered space $X = (X, \tau, \leq)$. Then $\Phi = QUC(X, \mathcal{V})$ is a maximal defining family for $X$ and $\mathcal{V} = \mathcal{V}_\Phi$.

**COROLLARY 3.5** Let $X$ be a $T_{3.5}$-ordered space. Then $\mathcal{V} \mapsto QUC(X, \mathcal{V})$ is bijective and order-preserving between the $T_0$, totally bounded quasi-uniformities which induce $X$ and the maximal defining families for $X$.

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