ON RADIiS OF STARLIKENESS AND CONVEXITY
FOR CONVOLUTIONS OF STARLIKE FUNCTIONS

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ABSTRACT. In this paper, we obtain the radiuses of univalence, starlikeness and convexity
for convolutions of starlike functions.

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1. INTRODUCTION

Let \( \mathcal{A} \) denote the class of functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) that are analytic in the unit
disc \( D = \{ z : |z| < 1 \} \), and let \( S \) denote the subclass of \( \mathcal{A} \) consisting of univalent functions. Let
\( S^* \) and \( K \) be the usual subclasses of \( S \) consisting of starlike and convex functions, respectively,
that is, \( S^* = \{ f : \text{Re}(zf'(z))/f(z) > 0 \} \) and \( K = \{ f : \text{Re}(1 + z f''(z)/f'(z)) > 0 \} \). The con-
volution or Hadamard product of two power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \)
is defined as the following power series \( (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \). Hadamard products have
many interesting properties and important applications, see [3] and [4]. It is well known that
if \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^* \), then \( z + \sum_{n=1}^{\infty} a_n z^n = \int_0^z \frac{f(t)}{t} \, dt \in K \).

Theorem A (see [1]). If \( f \in K \) and \( g \in K \) (\( g \in S^* \)), then \( f * g \in K \) (\( f * g \in S^* \)).

However, it is also known that if \( f \in S^* \) and \( g \in S^* \), \( f * g \) need not be in \( S^* \). Furthermore,
Sheil-Small in [2] showed that \( f * g \) need not be in \( S \) for \( f \in S^* \) and \( g \in S^* \).

2. MAIN RESULTS

Lemma 1. Let \( F(z) = z + \sum_{n=2}^{\infty} n^2 z^n \). Then \( F(z) \) is starlike in \( |z| < 2 - \sqrt{3} \approx 0.268 \).
The result is sharp.

Proof. Noting that
\[
F(z) = \frac{(z + 1)z}{(1 - z)^2}
\]
and differentiating logarithmically both sides of (1), we have
\[
\frac{z F'(z)}{F(z)} = \frac{z^2 + 4z + 1}{(1 + z)(1 - z)} = \frac{1 + z}{1 - z} - \frac{1}{1 + z} + \frac{1}{1 - z}.
\]
It follows from (2) that
\[
\text{Re} \left( \frac{z F'(z)}{F(z)} \right) \geq \frac{1 - r}{1 + r} - \frac{1}{1 - r} + \frac{1}{1 + r} = \frac{r^2 - 4r + 1}{(1 + r)(1 - r)},
\]
where \( r = |z| \). Thus, if \( |z| < 2 - \sqrt{3} \), then \( \text{Re}(z F'(z)/F(z)) > 0 \). So \( F(z) \) is starlike for
\( |z| < 2 - \sqrt{3} \). Since \( F'(-2 + \sqrt{3}) \neq 0 \), we know that the result is sharp.

Lemma 2. Let \( F(z) = z + \sum_{n=2}^{\infty} n^2 z^n \), then \( F(z) \) is convex in \( |z| < 5 - 2\sqrt{6} \approx 0.101 \).
The result is sharp.

Proof. Using (1), we have
\[
1 + \frac{z F''(z)}{F'(z)} = \frac{(1 + z)(z^2 + 10z + 1)}{(1 - z)(z^2 + 4z + 1)} = \frac{1 + z}{1 - z} + \frac{2}{z + 2 + \sqrt{3}} - \frac{2 - \sqrt{3}}{z + 2 - \sqrt{3}}.
\]
\[ \text{Re} \left( 1 + \frac{z F''(z)}{F'(z)} \right) \geq \frac{1 - r}{1 + r} + \frac{2}{2 + \sqrt{3} - r} - \frac{2 - \sqrt{3}}{2 + \sqrt{3} - r} = \frac{(1 - r)(r^2 - 10r + 1)}{(1 + r)(r^2 - 4r + 1)} \]

for \( r = |z| < 2 - \sqrt{3} \). Thus, we have \( \text{Re}(1 + z F''(z)/F'(z)) > 0 \) for \( |z| < 5 - 2\sqrt{6} \). Hence \( F(z) \) is convex for \( |z| < 5 - 2\sqrt{6} \). It is clear that the result is sharp.

**Theorem 1.** Let \( f \in S^* \) and \( g \in S^* \), then \( f \ast g \) is univalent and starlike for \( |z| < r_0 = 2 - \sqrt{3} \) and the constant \( r_0 \) cannot be replaced by any larger number.

**Proof.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^* \), \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^* \) and \( G(z) = f(z) * g(z) \). Then

\[ G(z) = (z + \sum_{n=2}^{\infty} a_n z^n) * (z + \sum_{n=2}^{\infty} b_n z^n) = (z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n) * (z + \sum_{n=2}^{\infty} \frac{b_n}{n} z^n). \]

We know that \( z + \sum_{n=2}^{\infty} a_n z^n \in K \) and \( z + \sum_{n=2}^{\infty} b_n z^n \in K \). By Theorem A, we get

\[ (z + \sum_{n=2}^{\infty} a_n z^n) * (z + \sum_{n=2}^{\infty} b_n z^n) \in K. \]

Now, let \( H(z) = (z + \sum_{n=2}^{\infty} a_n z^n) * (z + \sum_{n=2}^{\infty} b_n z^n) \), then \( H(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \). So that

\[ G(z) = (z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n) * H(z) = F(z) * H(z), \]

where \( F(z) = z + \sum_{n=2}^{\infty} a_n z^n \). By Lemma 1, we know that \( F(z) \) is starlike for \( |z| < r_0 = 2 - \sqrt{3} \). Hence \( F(r_0 z)/r_0 \in S^* \). Since \( H(z) \in K \), by Theorem A we have

\[ B(z) = (F(r_0 z)/r_0) * H(z) = z + \sum_{n=2}^{\infty} a_n b_n r_0^{n-1} z^n \in S^*. \]

Therefore, \( G(z) = r_0 B(z/r_0) \) is starlike for \( |z| < r_0 = 2 - \sqrt{3} \).

Finally, we show that \( r_0 \) cannot be replaced by any larger number. Taking \( \frac{z}{(1 - z)^2} \in S^* \), for \( G(z) = \frac{z}{(1 - z)^2} * \frac{z}{(1 - z)^2} = z + \sum_{n=2}^{\infty} n^2 z^n \), we have \( G'(-r_0) = 0 \). Thus, for any \( r > r_0 \), \( G(z) \) is not univalent for \( |z| < r \). This completes the proof of our theorem.

**Theorem 2.** Let \( f \in S^* \) and \( g \in S^* \), then \( f \ast g \) is convex for \( |z| < r_1 = 5 - 2\sqrt{6} \) and the constant \( r_1 \) cannot be replaced by any larger number.

**Proof.** By the method used in the proof of Theorem 1 and by using Lemma 2, we get Theorem 2 immediately and the sharpness of the result in Theorem 2 is obtained from (3).

**Remark.** The constant \( r_0 \) in Theorem 1 is usually referred to as the radius of univalence and starlikeness, while the constant \( r_1 \) in Theorem 2 is called the radius of convexity.

**REFERENCES**
