A CHARACTERIZATION OF B*-ALGEBRAS

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Abstract. A characterization of B*-algebras amongst all Banach algebras with bounded approximate identities is obtained.

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1. Introduction.

We recall that an approximate identity in a Banach algebra $A$ is a net $\{e_{\alpha} : \alpha \in I\}$ in $A$ where $I$ is a directed set such that $\lim_{\alpha} e_{\alpha} x = x = \lim_{\alpha} x e_{\alpha}$ for every $x$ in $A$. If there is a finite constant $M$ such that $\|e_{\alpha}\| \leq M$ for all $\alpha$, then the approximate identity is said to be bounded.

Let $A$ be a Banach algebra. For each $x$ in $A$, let

$$D_A(x) = \{f \in A' : \|f\| = 1 = f(x)\}.$$  

By a corollary of the Hahn–Banach theorem, $D_A(x)$ is non-empty. We denote $S(A) = \{x \in A : \|x\| = 1\}$.

For each $a \in A$, we call the set $V_A(a) = \{f(az) : f \in D_A(x), x \in S(A)\}$ the spatial numerical range of $a$.

We recall [5] that the relative numerical range of $a$ in $A$ with respect to $x \in A$, is defined as

$$\hat{V}_x(A, a) = \{f(az) : f \in D_A(x)\}.$$  

Thus we see that $V_A(a) = \bigcup \{\hat{V}_x(A, a) : x \in S(A)\}$, which is a bounded subset of the complex numbers bounded by $\|a\|$.

If $A$ has an approximate identity of norm less than or equal to one then $A$ can embedded, isometrically and isomorphically, in a unital Banach algebra $A^+$ in such a way that for each $a$ in $A$

$$V(A^+, a) = \overline{\mathcal{O}} V_A(a),$$

where $V(A^+, a) = \{f(a) : f \in (A^+)', \|f\| = 1 = f(a) = \|a\|\}$. For details see [4], Theorem 2.3.
An element $h$ of a Banach algebra $A$ is said to be Hermitian if $V_{A}(a) \subset R$. We denote by $H(A)$ the set of all Hermitian elements of $A$. A $B^{*}$-algebra is a Banach algebra $A$ with an involution, $a \rightarrow a^{*}$ satisfying the following conditions:

1. $(a + b)^{*} = a^{*} + b^{*}$;
2. $(ab)^{*} = b^{*}a^{*}$;
3. $(aa)^{*} = a\overline{a}$;
4. $a^{**} = a$; and
5. $|a^{*}a| = |a|^{2}$

for all $a, b \in A$ and $\alpha \in C$.

An element $a$ in a $B^{*}$-algebra is said to be self-adjoint if $a = a^{*}$. The set of all self-adjoint elements will be denoted by $S(A)$. Each element $a \in A$ can be written uniquely in the form $a = h + ik$ where $h, k \in S(A)$. Some of the well known properties of $S(A)$ are the following:

(a) The set $S(A)$ is a real partially ordered Banach space,
(b) each of its elements has real spectrum,
(c) if $h, k \in S(A)$ then $i(hk - kh) \in S(A)$, and
(d) for each $h \in S(A)$, the spectral radius $\rho(h) = ||h||$.

It is clear that the set of Hermitian elements, $H(A)$, of a Banach algebra with a bounded approximate identity of norm less than or equal to one has many of the properties of $S(A)$ in a $B^{*}$-algebra.

In this note we prove that in an arbitrary $B^{*}$-algebra $A$, $H(A) = S(A)$ in Theorem 2.1. This results mimics a result by Bohnenblust and Karlin [2].

In [8], Vidav has shown that a unital Banach algebra $A$ with the following conditions:

1. $A = H(A) + iH(A)$;
2. for each $h$ in $H(A)$ there exists $h_{1}, h_{2}$ in $H(A)$ such that $h_{1} + ih_{2} = h^{2}$ and $h_{1}h_{2} = h_{2}h_{1}$

is a $B^{*}$-algebra with Vidav-involution. Combining the results of Vidav [8], Berkson [1], and Glickfeld [6] we obtain the result that if $A$ is a unital Banach algebra such that $A = H(A) + iH(A)$ then $A$ is a $B^{*}$-algebra under the Vidav-involution. Here, we extend this result to the nonunital case in the form of Lemma 3.1.

Finally, combining the results of Theorem 2.1 and Lemma 3.1 we have a characterization of $B^{*}$-algebras with bounded approximate identities.

2. Some Results.

We now prove the following theorem.

**Theorem 2.1** Let $A$ be a $B^{*}$-algebra with a bounded approximate identity of norm less than or equal to one. An element of $A$ is Hermitian if and only if it is self-adjoint.

**Proof.** Case 1. Suppose that $A$ has a unit element 1. Let $f \in D_{A}(1)$. Then it is known that such a functional has the property that $f(h^{*}) = \overline{f(h)}$, for every $h \in A$. Thus if $h$ is a self-adjoint element of $A$, $f(h) = f(h^{*}) = \overline{f(h)}$ and hence $f(h)$ is real for all $f \in D_{A}(1)$. Hence, $S(A) \subseteq H(A)$.

Case 2. If $A$ has no identity element then it will have an approximate identity of norm less than or equal to one. Also, with the involution defined by $(a, \alpha)^{*} = (\overline{a^{*}}, \alpha)$ for $(a, \alpha) \in A^{+}$, and
by Theorem 2.3 in [4], \( A^+ \) becomes a unital \( B^* \)-algebra containing as a sub-\( B^* \)-algebra, ([3], 1.3.8).

Let \( h \) be a self-adjoint element of \( A \). Then \( (h, 0) \) is self-adjoint and hence Hermitian in the unital \( B^* \)-algebra \( A^+ \). Hence \( h \in H(A) \). We have therefore for any \( B^* \)-algebra, \( S(A) \subseteq H(A) \).

Suppose conversely that \( h \in H(A) \). Then for \( h_1 \) and \( h_2 \) in \( S(A) \), \( h = h_1 + ih_2 \). This implies that \( \nu(h_2) = 0 \) (where \( \nu(x) = \sup \{|\lambda| : \lambda \in V_A(x)\} \) and is called numerical radius of \( x \) in \( A \)) and hence \( h_2 = 0 \). Thus \( h = h_1 \) so that \( h \) is self-adjoint. That is \( H(A) \subseteq S(A) \) and hence the theorem.

**Remark 2.1** The above theorem shows that in a \( B^* \)-algebra the Hermitian elements generate the whole algebra in the sense that each element \( a \) may be written in the form \( a = h_1 + ih_2 \) with \( h_1 \) and \( h_2 \) in \( H(A) \). In an arbitrary Banach algebra \( A \) this is not true. We therefore consider the set \( J(A) = H(A) + iH(A) \). Since \( H(A) \) is a real space it follows that \( J(A) \) is a complex linear space. If \( A \) has no unit element then by Theorem 2.3, [4], \( J(A) \times C = J(A^+) \). We define a map \( a \rightarrow a^* \) from \( J(A) \) into itself by

\[(h_1 + ih_2)^* = h_1 - ih_2, \text{ for all } h_1, \ h_2 \in H(A).\]

The linear map \( a \rightarrow a^* \) is known as the Vidav-involution on \( J(A) \).

**Remark 2.2** If \( A \) has no unit element then it is a simple matter to verify that the Vidav-involution on \( J(A^+) \) is an extension of the Vidav-involution on \( J(A) \). The space \( J(A) \) is a complex Banach space and \( a \rightarrow a^* \) is a continuous linear involution on \( J(A) \). In general, the Banach space \( J(A) \) is not an algebra, and if \( J(A) \) is an algebra under some conditions, then the Vidav-involution has the additional property

\[(ab)^* = a^*b^*, \text{ for all } a, \ b \in J(A).\]

### 3. Characterization.

Vidav has shown in [8] that a unital Banach algebra \( A \) with the following conditions:

(V1) \( A = H(A) + iH(A) \),

(V2) for each \( h \) in \( H(A) \) there exists \( h_1, h_2 \) in \( H(A) \) such that \( h_1 + ih_2 = h^2 \) and \( h_1h_2 = h_2h_1 \),

is a \( B^* \)-algebra with Vidav-involution and a norm equivalent to the original norm on \( A \).

According to Palmer [7], the condition (V1) implies (V2). Also Berkson [1], Glickfeld [6], and Palmer [7] have shown that if (V1) is satisfied by the algebra \( A \) the equivalent norm by Vidav is equal to the original norm on \( A \). So by these results we have the result that if \( A \) is a unital Banach algebra satisfying (V1) then \( A \) is \( B^* \)-algebra under the Vidav-involution. The following lemma extends this result to the non-unital case.

**Lemma 3.1** Let \( A \) be a Banach algebra with a bounded approximate identity of norm less than or equal to one. Suppose that every \( a \) in \( A \) has the form \( a = h_1 + ih_2 \), for all \( h_1, h_2 \) in \( H(A) \). Then with the Vidav-involution, \( A \) is a \( B^* \)-algebra.

**Proof.** From Remark 2.1 we have that \( J(A^+) = J(A) \times C \). Since \( J(A) = A \) (by the hypothesis) we have \( J(A^+) = A^+ \). Therefore \( A^+ \) is a unital \( B^* \)-algebra under the Vidav-involution. Furthermore, \( A \) is a closed and self adjoint subalgebra of \( A^+ \), and is therefore a \( B^* \)-algebra under the Vidav-involution.
Finally, combining the results of Theorem 2.1 and Lemma 3.1 we have the following:

**Theorem 3.2** Let $A$ be a Banach algebra with a bounded approximate identity of norm less than or equal to one. Then $A$ is a $B^*$-algebra under some involution if and only if each element $a$ of $A$ can be written in the form $a = h_1 + ih_2$ where $h_1$ and $h_2$ are Hermitian elements of $A$.

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**References**


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