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COHOMOLOGY, DIMENSION AND LARGE RIEMANNIAN MANIFOLDS

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ABSTRACT. This paper surveys recent results on dimension and cohomology of the Higson corona of uniformly contractable manifolds.

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1. INTRODUCTION

There are many well-known conjectures and problems in the area of activity surrounding the Novikov Higher Signature Conjecture. Different branches of mathematics meet there such as differential geometry, algebraic topology, operator algebras, K-theory, geometric topology, geometric group theory The goal of this note is to demonstrate that dimension theory also comes into the picture

First we formulate two conjectures in the area the formulations of which is less technical. We recall that a manifold M^n is called aspherical if its homotopy groups $\pi_i(M)$ are trivial in dimensions ≥ 2 The following is a special case of the Strong Novikov conjecture [1].

GROMOV-LAWSON CONJECTURE. A closed aspherical manifold does not admit a Riemannian metric of a positive scalar curvature.

One of the strongest in the area is the following

BOREL CONJECTURE. Any two homotopy equivalent closed aspherical manifold are homeomorphic.

The standard approach in any problem about an aspherical manifold N^n consists of taking into consideration the universal cover M^n of N^n together with the action of the fundamental group $\Gamma = \pi_1(N^n)$. As it follows from Whitehead theorem, the universal cover of an aspherical manifold is contractible. Since every contractible manifold M^n crossed with the real line $\mathbf{R}$ is homeomorphic to euclidean space $\mathbf{R}^{n+1}$, without a big loss we may assume that M^n is homeomorphic to $\mathbf{R}^n$. So, the problem about an aspherical manifold is a problem about a Γ-periodic metric on $\mathbf{R}^n$. The main feature of a Γ-periodic metric in this case is that the metric space $M = \mathbf{R}^n$ is uniformly contractible. We recall that a metric space X, ρ is *uniformly contractible* if for every positive number R there is a bigger number S such that every ball $B_\rho(x, R)$ can be contracted to a point in $B_\rho(x, S)$.

The study of a non-compact space can be simplified by adding a corona to M making M compact Such a corona for metric spaces was introduced by Higson [2].

DEFINITION. If M is a space and $\phi : M \to \mathbf{C}$ is a continuous function, define $V_r(\phi) : M \to \mathbf{R}^+$ by

$$V_r(\phi) = \sup\{|\phi(y) - \phi(x)| : y \in B_r(x)\}.$$

Then $C_h(M)$ is the space of all bounded continuous functions $\phi : M \to \mathbf{C}$ so that for each $r > 0$, $V_r(\phi) \to 0$ at infinity. Lemma 5.3 of [3] proves that $C_h(M)$ is a C^*-algebra, so it makes sense to define the *Higson compactification*, $\bar{M}$ of M to be the maximal ideal space of $C_h(M)$. Then the remainder of the Higson compactification $\nu M = \bar{M} \backslash M$ is called the *Higson corona*

The Higson corona was introduced for an analysis of the index theorem for noncompact manifolds J Roe demonstrated in [3] that the Higson corona can be used for characterization of the notion of *hypereuclidean space* which is due to Gromov [4]:

A manifold M^n is hypereuclidean if there is a map $f : \nu M^n \to S^{n-1}$ of the degree one of the Higson corona onto $n-1$-sphere

The degree of f in this case is the degree of the homomorphism $\partial f^* : H^{n-1}(S^{n-1}) = \mathbf{Z} \to \mathbf{Z} = H_c^n(M^n)$.

Let $\tilde{M}^n$ be a compactification of M^n. An action of Γ on M^n by isometries is called *small at infinity* with respect to a compactification $\tilde{M}^n$ if for every point $x \in \tilde{M}^n \backslash M^n$ and for every compact set $C \subset M^n$ for every neighborhood U of x there is a smaller neighborhood V such if $g(C) \cup V \neq \emptyset$ for some $g \in \Gamma$, then $g(C) \subset U$. Carlsson and Pedersen proved in [5] (see also [6]) that if the universal cover M^n admits an acyclic compactification with a small action of Γ at infinity, then the Novikov conjecture holds for the group Γ.

PROPOSITION. Let $\tilde{M}^n$ be a compactification of M^n An action of Γ on M^n is small at infinity with respect to a compactification $\tilde{M}^n$ if and only if there is a map $g : \bar{M}^n \to \tilde{M}^n$ of the Higson compactification, which is the identity of M^n.

PROOF. The 'only if' case was proven in [6, Proposition 3.1].

Let $g : \bar{M}^n \to \tilde{M}^n$ be that map. Take a point $x \in \tilde{M}^n \backslash M^n$, take a compact set $C \subset M^n$ and a neighborhood U containing x. There is a compact metric space $\check{M}^n, \rho$ and a map $q : \tilde{M}^n \to \check{M}^n$ such that the restriction $q|_{M^n} = id_{M^n}$ and there is an open set $U', q(x) \in U'$ in $\check{M}^n$ with $q^{-1}(U') \subset U$ Let $d = dist_\rho(q(x), \check{M}^n \backslash U')$. By Proposition 1 of [7] $lim_{y\to\infty} diam(B(y,R)) = 0$ for any R, here $B(y,R)$ is the ball in the metric space M^n of radius R and centered at y Since Γ acts on M^n by isometries, the diameter of $g(C)$ is equal to the diameter of C for all $g \in \Gamma$ Hence there is a neighborhood O of the boundary $\check{M}^n \backslash M^n$ such that $diam\, g(C) < \frac{d}{2}$ whenever $g(C) \cup O \neq \emptyset$ Define $V = q^{-1}(IntB_\rho(q(x), \frac{d}{2}) \cap O)$.

In [8] it is proven that the Gromov-Lawson conjecture holds for manifolds N^n with the hypereuclidean universal cover M^n. This result makes the following conjecture natural.

GROMOV'S CONJECTURE [4] *Every uniformly contractible manifold is hypereuclidean*

Unfortunately this version of Gromov's conjecture is not correct.

THEOREM 1 [9]. There is a uniformly contractible metric on $\mathbf{R}^8$ which is not hypereuclidean

This counterexample is based on the strange phenomenon in dimension theory: cohomological and covering dimension may disagree for infinite-dimensional spaces [10].

Note that the rational version of Gromov's conjecture still may be correct. To avoid our counterexample in the integral case one should restrict oneself to uniformly contractible manifolds with bounded geometry [11]. Any of these two modified versions would be sufficient to derive the Gromov-Lawson conjecture. According to [12, Theorem 3.1] the rational version of Gromov's conjecture is equivalent to the following:

WEINBERGER CONJECTURE [3]. *For every uniformly contractible manifold X the boundary homomorphism $\partial H^{n-1}(\nu X; \mathbf{Q}) \to H_c^n(\mathbf{R}^n; \mathbf{Q})$* is an epimorphism

The Weinberger conjecture can be easily verified for the standard euclidean spaces $\mathbf{R}^n$ and for the hyperbolic spaces $\mathbf{H}^n$ [6]. The stronger conjecture (see [3]) was that the Higson compactification of a

uniformly contractible manifold is acyclic This conjecture was disproved by J Keesling [13] who showed that the 1-dimensional cohomology group of the Higson compactification of the euclidean space is nontrivial As it can easily be seen, to derive the Weinberger conjecture it is sufficient to have trivial only n-dimensional cohomology group. It turns out to be that this also is not true even for the euclidean space

THEOREM 2 [12]. $H^n(\overline{\mathbf{R}^n}; \mathbf{Q}) \neq 0$

Nevertheless in the hyperbolic case there is an acyclicity theorem [12].

In the example of Theorem 1 the dimension of the Higson corona νM^8 is infinite [6] Perhaps this is the main obstruction for a uniformly contractible manifold to be hyperspherical.

PROBLEM. Does the Gromov-Lawson Conjecture (the Novikov Conjecture) hold for manifolds N^n with finite dimension of the Higson corona of the universal cover. $\dim \nu M^n < \infty$?

The importance of this question is supported by the following two results.

THEOREM 3 [14] If $asdimM^n < \infty$ then the Gromov-Lawson Conjecture (and the Novikov Conjecture) holds for M^n

We recall that $asdim$ stands for the asymptotic dimension introduced by Gromov [15]. By the definition $asdimM^n \leq n$ if for every positive $R > 0$ there exists a uniformly bounded covering $\mathcal{U}$ such that every R-ball $B(x, R)$ in M^n intersects no more than $n+1$ elements of $\mathcal{U}$.

THEOREM 4 [6]. For every proper metric space M there is the inequality $\dim \nu M \leq asdimM$

In the conclusion we formulate the classical Novikov conjecture.

NOVIKOV CONJECTURE. Let G_n^k be the Grassmanian space of k-dimensional oriented vector subspaces in n-space with the natural topology. There is the natural imbedding $G_n^k \subset G_{n+1}^k$. Then one can define the space $G_\infty^k = \lim_{\rightarrow} G_n^k$. The natural imbedding $G_\infty^k \subset G_\infty^{k+1}$ leads to the definition of the space $BO = G_\infty^\infty = \lim_{\rightarrow} G_\infty^k$. The tangent bundle of an n-dimensional manifold N can be obtained as the pull-back from the natural n-bundle over the space G_∞^n. Let $f_\tau : N \to BO$ be a map which induces the tangent bundle on N. The cohomology ring $H^*(BO; \mathbf{Q})$ is a polynomial ring generated by some elements $a_i \in H^{4i}(BO; \mathbf{Q})$. The rational Pontryagin classes of a manifold N are the elements $p_i = f^*(a_i) \in H^{4i}(BO; \mathbf{Q})$. Novikov [16] proved that the rational Pontryagin classes are topological invariants. It was known that they are not homotopy invariants. Hirzebruch found polynomial $L_k(p_1, ..., p_k) \in H^{4i}(N; \mathbf{Q})$ which do not depend on N and such that the signature of every closed (oriented) $4k$ manifold N can be defined as the value of L_k on the fundamental class of N. Note that the signature is homotopy invariant and even more, it is bordism invariant. For non-simply connected manifolds Novikov defined the higher signature as follows Let Γ be the fundamental group of a closed oriented manifold N, let $g : N \to B\Gamma = K(\Gamma, 1)$ be a map classifying the universal cover of N and let $b \in H^*(K(\Gamma, 1); \mathbf{Q})$. Then he defines the b-signature as $sign_b(N) = \langle L_k \cap g^*(b), [N] \rangle$, here $4k + dim(b) = dimN$.

CONJECTURE. Let $h : N \to M$ be an orientation preserving homotopy equivalence between two close oriented manifolds, then $sign_b(N) = sign_b(M)$ for any $b \in H^*(K(\Gamma, 1); \mathbf{Q})$.

We say that the Novikov conjecture holds for a group Γ if it holds for every manifold with the fundamental group Γ. It is known that the Novikov conjecture holds for Γ if and only if the certain homomorphism from the surgery exact sequence [17],[18],[19], called *the assembly* map [20]

$$A : H_*(K(\Gamma, 1)\mathbf{Q}) \to L_*(\mathbf{Z}[\Gamma]) \otimes \mathbf{Q}$$

is an injection. The Novikov conjecture is verified for abelian groups, hyperbolic groups, CAT(0)-groups, discrete subgroups of connected Lie groups and others [4],[6],[16],[21],[22],[23],[24],[25],[26]

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