INEQUALITIES VIA LAGRANGE MULTIPLIERS

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ABSTRACT. An easy method is obtained to prove many inequalities using Lagrange multipliers.

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1. INTRODUCTION

Let us assume that $d_1, \dots, d_n$ are unit perpendicular vectors in an n -dimensional space X . In particular d_1, d_2 , and d_3 are the unit perpendicular vectors i, j , and k in the 3-dimensional space. Any vector v in X is usually uniquely written in the form

$$v = \sum_{i=1}^n \lambda_i d_i$$

for scalars λ_i . We define

$$\nabla f(x_1, \dots, x_n) = \sum_{i=1}^n f_{x_i}(x_1, \dots, x_n) d_i, \quad f_x = \frac{\partial}{\partial x}.$$

Kapur and Kumar (1986), have used the principle of dynamic programming to prove major inequalities due to Shannon, Renyi, and Holder, see [1]. In this note we give a new method using Lagrange multipliers.

2. SHANNON'S INEQUALITY

THEOREM 2.1. Given $\sum_{i=1}^n p_i = a$, $\sum_{i=1}^n q_i = b$, then

$$a \ln(a/b) \leq \sum_{i=1}^n p_i \ln(p_i/q_i), \quad p_i, q_i \geq 0.$$

The equality holds iff $p_i = q_i$ for each i .

PROOF. Let the q_i 's and a be fixed; set

$$f(p_1, \dots, p_n) = \sum_{i=1}^n p_i \ln(p_i/q_i); \quad p_i, q_i \geq 0,$$

we aim to minimize f subject to the constraint

$$g(p_1, \dots, p_n) = \sum_{i=1}^n p_i - a = 0.$$

There is a minimum achieved where $\nabla f = \lambda \nabla g$ because g is linear and f is convex, since its second order partials are all non-negative

$$\begin{aligned}\nabla f = \lambda \nabla g &\Rightarrow \sum_{i=1}^n \{1 + \ln(p_i/q_i)\} d_i = \lambda \sum_{i=1}^n d_i \\ &\Rightarrow 1 + \ln(p_i/q_i) = \lambda \\ &\Rightarrow \frac{p_1}{q_1} = \frac{p_2}{q_2} = \dots = \frac{p_n}{q_n} = \frac{\sum a_i}{\sum b_i} = \frac{a}{b}.\end{aligned}$$

Therefore

$$\min \sum_{i=1}^n p_i \ln(p_i/q_i) = \ln(a/b) \sum_{i=1}^n p_i = a \ln(a/b),$$

or

$$a \ln(a/b) \leq \sum_{i=1}^n p_i \ln(p_i/q_i).$$

If $a = b = 1$, we get Shannon's inequality

$$\sum_{i=1}^n p_i \ln(p_i/q_i) \geq 0 \quad \text{and} \quad \sum_{i=1}^n p_i \ln(p_i/q_i) = 0 \quad \text{iff} \quad p_i = q_i \quad \text{for each } i.$$

3. RENYI'S INEQUALITY

THEOREM 3.1. Given $\sum_{i=1}^n a_i = a$, $\sum_{i=1}^n b_i = b$, then

$$\frac{1}{\alpha - 1} (a^\alpha b^{1-\alpha} - a) \leq \sum_{i=1}^n \frac{1}{\alpha - 1} (p_i^\alpha q_i^{1-\alpha} - p_i), \quad p_i, q_i \geq 0, 0 < \alpha \neq 1.$$

The equality holds iff $p_i = q_i$ for each i .

PROOF. Let the q_i 's and a be fixed and write

$$f(p_1, \dots, p_n) = \sum_{i=1}^n \frac{1}{\alpha - 1} p_i^\alpha q_i^{1-\alpha}, \quad g(p_1, \dots, p_n) = \sum_{i=1}^n p_i - \alpha = 0$$

$$\begin{aligned}\nabla f = \lambda \nabla g &\Rightarrow \sum_{i=1}^n \frac{\alpha}{\alpha - 1} p_i^{\alpha-1} q_i^{1-\alpha} d_i = \lambda \sum_{i=1}^n d_i \\ &\Rightarrow (p_i/q_i)^{\alpha-1} = \lambda \left(\frac{\alpha - 1}{\alpha} \right) \\ &\Rightarrow \frac{p_1}{q_1} = \dots = \frac{p_n}{q_n} = \frac{a}{b} \\ &\Rightarrow \min f(p_1, \dots, p_n) = \frac{1}{\alpha - 1} a^\alpha b^{1-\alpha},\end{aligned}$$

by the convexity of f and linearity of g . Hence

$$\frac{1}{\alpha - 1} a^\alpha b^{1-\alpha} \leq \sum_{i=1}^n \frac{1}{\alpha - 1} p_i^\alpha q_i^{1-\alpha}.$$

If $a = b = 1$, we get Renyi's inequality

$$\frac{1}{\alpha - 1} \left(\sum_{i=1}^n p_i^\alpha q_i^{1-\alpha} - 1 \right) \geq 0.$$

4. HOLDER'S INEQUALITY

THEOREM 4.1. Given $\sum_{i=1}^n a_i^p = A$, $\sum_{i=1}^n b_i^q = B$, $\sum_{i=1}^n a_i b_i = C$, $a_i, b_i \geq 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

then

$$C \leq A^{1/p} B^{1/q}. \quad (4.1)$$

PROOF. This follows from Renyi's inequality, taking $\alpha = 1/p$, $a_i = p_i^p$, $b_i = q_i^q$, or, we prove the result directly as follows:

let the a_i 's and C be fixed and write

$$f(b_1, \dots, b_n) = A^{q/p} \sum_{i=1}^n b_i^q, g(b_1, \dots, b_n) = \sum_{i=1}^n a_i b_i - C = 0$$

$$\nabla f = \lambda \nabla g \Rightarrow q A^{q/p} \sum_{i=1}^n b_i^{q-1} d_i = \lambda \sum_{i=1}^n a_i d_i$$

$$\Rightarrow A^{q/p} b_i^{q-1} = (\lambda/q) a_i \quad (4.2)$$

$$(4.2) \Rightarrow A^{q/p} = (\lambda/q) C, \quad (4.3)$$

and

$$A^q B = (\lambda/q) A, \quad \text{as } p(q-1) = q \quad (4.4)$$

$$(4.3) \& (4.4) \Rightarrow \lambda/q = C^{q-1}.$$

Therefore, by the convexity of f and linearity of g ,

$$\min(A^{q/p} B) = C^q,$$

or

$$C \leq A^{1/p} B^{1/q}.$$

5. GENERALIZATIONS OF HOLDER'S INEQUALITY

THEOREM 5.1. Given $\sum_{i=1}^n a_i^p = A$, $\sum_{i=1}^n b_i^q = B$, $\sum_{i=1}^n c_i^r = C$, and $\sum_{i=1}^n a_i b_i c_i = D$, $a_i, b_i, c_i \geq 0$,

$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then

$$D \leq A^{1/p} B^{1/q} C^{1/r}.$$

PROOF. This follows by an easy application of Holder's inequality:

$$\begin{aligned} \sum_{i=1}^n a_i b_i c_i &\leq \left[\sum_{i=1}^n (a_i b_i)^{\frac{r}{r-1}} \right]^{1-\frac{1}{r}} C^{\frac{1}{r}} \\ &= \left[\sum_{i=1}^n (a_i b_i)^{\frac{pq}{p+q}} \right]^{\frac{1}{p} + \frac{1}{q}} C^{\frac{1}{r}} \\ &\leq \left[\sum_{i=1}^n \left(a_i^{\frac{pq}{p+q}} \right)^{\frac{p+q}{q}} \right]^{\frac{p}{p+q} \left(\frac{p+q}{pq} \right)} \left[\sum_{i=1}^n \left(b_i^{\frac{pq}{p+q}} \right)^{\frac{p+q}{p}} \right]^{\frac{p}{p+q} \left(\frac{p+q}{pq} \right)} C^{\frac{1}{r}} \\ &= A^{\frac{1}{p}} B^{\frac{1}{q}} C^{\frac{1}{r}}. \end{aligned}$$

6. MINKOWSKI'S INEQUALITY

THEOREM 6.1. Given $\sum_{i=1}^n a_i^p = A$, $\sum_{i=1}^n b_i^p = B$, and $\sum_{i=1}^n (a_i + b_i)^p = C$, $a_i, b_i \geq 0$, $p \geq 1$, then

$$C^{\frac{1}{p}} \leq A^{\frac{1}{p}} + B^{\frac{1}{p}}.$$

PROOF. Let the b_i 's and A be fixed and write

$$f(a_1, \dots, a_n) = \sum_{i=1}^n (a_i + b_i)^p, \quad g(a_1, \dots, a_n) = \sum_{i=1}^n a_i^p - A = 0$$

$$\begin{aligned} \nabla f &= \mu \nabla g \Rightarrow \sum_{i=1}^n p(a_i + b_i)^{p-1} d_i = \mu \sum_{i=1}^n p a_i^{p-1} d_i \\ &\Rightarrow (a_i + b_i)^{p-1} = \mu a_i^{p-1} \\ &\Rightarrow \frac{b_1}{a_1} = \dots = \frac{b_n}{a_n} = C. \end{aligned}$$

Therefore,

$$\begin{aligned} \max C^{\frac{1}{p}} &= \left[\sum_{i=1}^n (a_i + c a_i)^p \right]^{\frac{1}{p}} \\ &= (1 + c) A^{\frac{1}{p}} \\ &= A^{\frac{1}{p}} + c A^{\frac{1}{p}} \\ &= A^{\frac{1}{p}} + B^{\frac{1}{p}}, \end{aligned}$$

or

$$C^{\frac{1}{p}} \leq A^{\frac{1}{p}} + B^{\frac{1}{p}}.$$

7. ARITHMETIC-GEOMETRIC-MEAN INEQUALITY

THEOREM 7.1.

$$\left(\prod_{i=1}^n x_i \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^n x_i.$$

PROOF. Write

$$f(x_1, \dots, x_n) = x_1 x_2 \dots x_n = y, \quad g(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n x_i - C = 0.$$

Let C be fixed, we have

$$\begin{aligned} \nabla f &= \mu \nabla g \Rightarrow \sum_{i=1}^n \frac{y}{x_i} d_i = \frac{\mu}{n} \sum_{i=1}^n d_i \\ &\Rightarrow x_i = \frac{n}{\mu} y \\ &\Rightarrow C = \frac{n}{\mu} y. \end{aligned}$$

Therefore

$$\max y^{\frac{1}{n}} = \frac{n}{\mu} y = C,$$

or

$$\left(\prod_{i=1}^n x_i \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^n x_i.$$

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