A STUDY OF SOME NEW ABSOLUTE SUMMABILITY METHODS

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ABSTRACT. In this note we introduce a new method of absolute summability. A general theorem is given. Several results are also deduced.

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1. INTRODUCTION.

Let $\sum a_n$ be an infinite series with partial sums s_n . Let σ_n^δ and η_n^δ denotes the n th Cesaro mean of order δ ($\delta > -1$) of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series $\sum a_n$ is said to be summable $|C, \delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty ,$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1} |\eta_n^\delta|^k < \infty .$$

Let $\{p_n\}$ be a sequence of real or complex constants with

$$P_n = p_0 + p_1 + \dots + p_n , \quad P_{-1} = p_{-1} = 0 .$$

The series $\sum a_n$ is said to be summable $|N, p_n|$, if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty , \tag{1}$$

where

$$t_n = P_n^{-1} \sum_{v=0}^n p_{n-v} s_v \quad (t_{-1}=0) .$$

We write $p = \{p_n\}$ and

$$M = \{p : p_n > 0 \text{ \& } p_{n+1}/p_n \leq p_{n+2}/p_{n+1}, n = 0, 1, \dots\}$$

It is known that for $p \in M$, (1) holds if and only if (Das [4])

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{v=1}^n p_{n-v} v a_v \right| < \infty .$$

DEFINITION 1 (Sulaiman [5]). For $p \in M$, we say that $\sum a_n$ is summable $|N, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{n P_n^k} \left| \sum_{v=1}^n p_{n-v} v a_v \right|^k < \infty.$$

In the special case in which $p_n = A_n^{r-1}$, $r > -1$, where A_n^r is the coefficient of x^n in the power series expansion of $(1-x)^{-r-1}$ for $|x| < 1$, $|N, p_n|_k$ summability reduces to $|C, r|_k$ summability.

The series $\sum a_n$ is said to be summable $|R, p_n|_k$, $|\overline{N}, p_n|_k$, $k \geq 1$ (Bor [2] & [1]), if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty, \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_n - T_{n-1}|^k < \infty,$$

respectively, where

$$T_n = P_n^{-1} \sum_{v=0}^n p_v s_v.$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $|R, p_n|_k$, $|\overline{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|R, p_n|$) summability.

We set

$$\begin{aligned} Q_n &= q_0 + q_1 + \dots + q_n, \quad q_{-1} = Q_{-1} = 0. \\ U_n &= u_0 + u_1 + \dots + u_n, \quad u_{-1} = U_{-1} = 0. \\ R_n &= p_0 q_n + p_1 q_{n-1} + \dots + p_n q_0 \\ \Delta f_n &= f_n - f_{n+1} \end{aligned}$$

We assume $\{\phi_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive real constants. Here we give the following new definition.

DEFINITION 2. Let $\{p_n\}$, $\{q_n\}$ be sequences of positive real constants such that $q \in M$. We say that $\sum a_n$ is summable $|N, R_n, \phi_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \phi_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^n P_{v-1} q_{n-v} a_v \right|^k < \infty.$$

DEFINITION 3 (Sulaiman [6]). The series $\sum a_n$ is said to be summable $|\overline{N}, p_n, \phi_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \phi_n^{k-1} |T_n - T_{n-1}|^k < \infty.$$

2. LEMMAS

LEMMA 1. Let $\{p_n\}$, $\{q_n\}$, and $\{u_n\}$ be sequences of positive real constants such that $q \in M$, $\{\alpha_n^{1-1/k} p_n / P_n R_{n-1}\}$ nonincreasing for $q_n \neq c$. Let T_n denote the $(\overline{N}, u_n)$ -mean of the series $\sum a_n$. Let $\{\epsilon_n\}$ be a sequence of constants and write $\beta_n^{1-1/k} \Delta T_{n-1} = \Delta_n$. If

$$\sum_{n=v+1}^{m+1} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}} q_{n-v-1} = O \left\{ \frac{\alpha_v^{k-1} p_v^{k-1}}{P_v^k} \right\},$$

$$\sum_{n=1}^{\infty} \left(\frac{\alpha_n}{\beta_n} \right)^k \left(\frac{p_n}{P_n} \right)^k \left(\frac{P_{n-1}}{R_{n-1}} \right)^k \left(\frac{U_n}{u_n} \right)^k |\epsilon_n|^k |\Delta_n|^k < \infty,$$

$$\sum_{n=1}^{\infty} \left(\frac{\alpha_n}{\beta_n} \right)^{k-1} |\epsilon_n|^k |\Delta_n|^k < \infty,$$

and

$$\sum_{n=1}^{\infty} \left(\frac{\alpha_n}{\beta_n} \right)^{k-1} \left(\frac{U_{n-1}}{u_n} \right)^k |\Delta \epsilon_n|^k |\Delta_n|^k < \infty$$

then the series $\sum a_n \epsilon_n$ is summable $|N, R_n, \alpha_n|_k$, $k \geq 1$.

LEMMA 2 (Sulaiman [7]) Let $q \in M$. Then for $0 < r \leq 1$,

$$\sum_{n=v+1}^{\infty} \frac{q_{n-v-1}}{n^r Q_{n-1}} = O(v^{-r}).$$

LEMMA 3 (Bor [2]). Let $k > 1$ and $A = (a_{nv})$ be an infinite matrix. In order that $A \in (l^k; l^k)$, it is necessary that

$$a_{nv} = O(1) \quad (\text{all } n, v) \quad (3)$$

Proof of Lemma 1. Write

$$\tau_n = \sum_{v=1}^n P_{v-1} q_{n-v} a_v \epsilon_v.$$

Since

$$T_n = U_n^{-1} \sum_{v=0}^n u_v \sum_{r=0}^v a_r = U_n^{-1} \sum_{v=0}^n (U_n - U_{v-1}) a_v,$$

then

$$- \Delta T_{n-1} = \frac{u_n}{U_n U_{n-1}} \sum_{v=1}^n U_{v-1} a_v.$$

By Abel's transformation,

$$\begin{aligned} \tau_n &= \sum_{v=1}^n U_{v-1} a_v (P_{v-1} q_{n-v} U_{v-1}^{-1} \epsilon_v) \\ &= \sum_{v=1}^{n-1} \left(\sum_{r=1}^v U_{r-1} a_r \right) \Delta (P_{v-1} q_{n-v} U_{v-1}^{-1} \epsilon_v) + \left(\sum_{r=1}^n U_{r-1} a_r \right) P_{n-1} q_0 U_{n-1}^{-1} \epsilon_n \\ &= \sum_{v=1}^{n-1} \left\{ - \frac{U_{v-1} U_v}{u_v} \Delta T_{v-1} \right\} \left\{ P_{v-1} \Delta_v q_{n-v} U_{v-1}^{-1} \epsilon_v + P_{v-1} q_{n-v-1} \frac{u_v}{U_{v-1} U_v} \epsilon_v \right. \\ &\quad \left. - P_v q_{n-v-1} U_v^{-1} \epsilon_v + P_v q_{n-v-1} U_v^{-1} \Delta \epsilon_v \right\} - P_{n-1} q_0 U_n u_n^{-1} \epsilon_n \Delta T_{n-1} \\ &= \sum_{v=1}^{n-1} \left\{ - P_{v-1} \Delta q_{n-v} \frac{U_v}{u_v} \epsilon_v \Delta T_{v-1} - P_{v-1} q_{n-v-1} \epsilon_v \Delta T_{v-1} + P_v q_{n-v-1} \frac{U_{v-1}}{u_v} \right. \\ &\quad \left. \times \epsilon_v \Delta T_{v-1} - P_v q_{n-v-1} \frac{U_{v-1}}{u_v} \Delta \epsilon_v \Delta T_{v-1} \right\} - P_{n-1} q_0 \frac{U_n}{u_n} \epsilon_n \Delta T_{n-1} \end{aligned} \quad (2)$$

$$= \tau_{n,1} + \tau_{n,2} + \tau_{n,3} + \tau_{n,4} + \tau_{n,5}, \quad \text{say}$$

In order to prove the lemma, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \alpha_n^{k-1} \left| \frac{P_n}{P_n R_{n-1}} \tau_{n,r} \right|^k < \infty, \quad r = 1, 2, 3, 4, 5.$$

Applying Hölder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \tau_{n,1} \right|^k &= \sum_{n=2}^{m+1} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} P_{v-1} \Delta_v q_{n-v} \frac{U_v}{u_v} \epsilon_v \Delta T_{v-1} \right|^k \\
&\leq \sum_{n=2}^{m+1} \alpha_n^{k-1} \left(\frac{p_n}{P_n R_{n-1}} \right)^k \sum_{v=1}^{n-1} P_{v-1}^k |\Delta_v q_{n-v}| \left(\frac{U_v}{u_v} \right)^k |\epsilon_v|^k |\Delta T_{v-1}|^k \\
&\quad \times \left\{ \sum_{v=1}^{n-1} |\Delta_v q_{n-v}| \right\}^{k-1} \\
&= 0(1) \sum_{v=1}^m P_{v-1}^k \left(\frac{U_v}{u_v} \right)^k |\epsilon_v|^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}^k} |\Delta_v q_{n-v}| \\
&= 0(1) \sum_{v=1}^m \left(\frac{\alpha_v}{\beta_v} \right)^{k-1} \left(\frac{p_v}{P_v} \right)^k \left(\frac{P_{v-1}}{R_{v-1}} \right)^k \left(\frac{U_v}{u_v} \right)^k |\epsilon_v|^k |\Delta_v|^k
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \tau_{n,2} \right|^k &= \sum_{n=2}^{m+1} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v-1}}{p_v} p_v q_{n-v-1} \epsilon_v \Delta T_{v-1} \right|^k \\
&\leq \sum_{n=2}^{m+1} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}^k} \sum_{v=1}^{n-1} \left(\frac{P_{v-1}}{p_v} \right)^k p_v q_{n-v-1} |\epsilon_v|^k |\Delta T_{v-1}|^k \left\{ \sum_{v=1}^{n-1} \frac{p_v q_{n-v-1}}{R_{n-1}} \right\}^{k-1} \\
&= 0(1) \sum_{v=1}^m \left(\frac{P_v}{p_v} \right)^k p_v |\epsilon_v|^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}^k} q_{n-v-1} \\
&= 0(1) \sum_{v=1}^m \left(\frac{\alpha_v}{\beta_v} \right)^{k-1} |\epsilon_v|^k |\Delta_v|^k
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \tau_{n,3} \right|^k &= \sum_{n=2}^{m+1} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} p_v q_{n-v-1} \frac{U_{v-1}}{u_v} v \epsilon_v \Delta T_{v-1} \right|^k \\
&\leq \sum_{n=2}^{m+1} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}^k} \sum_{v=1}^{n-1} p_v q_{n-v-1} \left(\frac{U_{v-1}}{u_v} \right)^k |\epsilon_v|^k |\Delta T_{v-1}|^k \\
&\quad \times \left\{ \sum_{v=1}^{n-1} \frac{p_v q_{n-v-1}}{R_{n-1}} \right\}^{k-1} \\
&= 0(1) \sum_{v=1}^m p_v \left(\frac{U_v}{u_v} \right)^k |\epsilon_v|^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}^k} q_{n-v-1} \\
&= 0(1) \sum_{v=1}^m \left(\frac{\alpha_v^{k-1}}{\beta_v} \right) \left(\frac{p_v}{P_v} \right)^k \left(\frac{P_{v-1}}{R_{v-1}} \right)^k \left(\frac{U_v}{u_v} \right)^k |\epsilon_v|^k |\Delta_v|^k
\end{aligned}$$

$$\begin{aligned}
\sum_{m=2}^{m+1} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \tau_{n,4} \right|^k &= \sum_{n=2}^{m+1} \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} P_v q_{n-v-1} \frac{U_{v-1}}{u_v} \Delta \epsilon_v \Delta T_{v-1} \right|^k \\
&\leq \sum_{n=2}^{m+1} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k p_v q_{n-v} \left(\frac{U_{v-1}}{u_v} \right)^k |\Delta \epsilon_v|^k |\Delta T_{v-1}|^k \\
&\quad \times \left\{ \sum_{v=1}^{n-1} \frac{p_v q_{n-v-1}}{R_{n-1}} \right\}^{k-1} \\
&= 0(1) \sum_{v=1}^m \left(\frac{P_v}{p_v} \right)^k p_v \left(\frac{U_{v-1}}{u_v} \right)^k |\Delta \epsilon_v|^k |\Delta T_{v-1}|^k \cdot \sum_{n=v+1}^{m+1} \frac{\alpha_n^{k-1} p_n}{P_n^k R_{n-1}} q_{n-v-1} \\
&= 0(1) \sum_{v=1}^m \left(\frac{\alpha_v}{\beta_v} \right)^{k-1} \left(\frac{\alpha_v^{k-1}}{\beta_v} \right) \left(\frac{U_v}{u_v} \right)^k |\Delta \epsilon_v|^k |\Delta v|^k
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^m \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} \tau_{n,5} \right|^k &= \sum_{n=1}^m \alpha_n^{k-1} \left| \frac{p_n}{P_n R_{n-1}} P_{n-1} q_0 \frac{U_n}{u_n} \epsilon_n \Delta T_{n-1} \right|^k \\
&= 0(1) \sum_{n=1}^m \left(\frac{\alpha_n}{\beta_n} \right)^{k-1} \left(\frac{p_n}{P_n} \right)^k \left(\frac{P_{n-1}}{R_{n-1}} \right)^k \left(\frac{U_n}{u_n} \right)^k |\epsilon_n|^k |\Delta n|^k.
\end{aligned}$$

This completes the proof of Lemma 1.

3. MAIN RESULT

THEOREM. Let $q \in M$ such that $\{\alpha_n^{1-1/k} p_n / P_n R_{n-1}\}$ nonincreasing for $q_n \neq c$. Let $P_n R_{n-1} u_n = 0(p_n P_{n-1} U_n)$, $\sum \alpha_n^{k-1} (p_n / P_n)^k$ divergent, and

$$\sum_{n=v}^{\infty} \frac{\alpha_n^{k-1} p_n^k}{P_n^k R_{n-1}} q_{n-v-1} = 0 \left\{ \frac{\alpha_v^{k-1} p_v^{k-1}}{P_v^k} \right\}.$$

Then the necessary and sufficient conditions that $\sum a_n \epsilon_n$ is summable $|N, R_n, \alpha_n|_k$ whenever $\sum a_n$ is summable $|\overline{N}, p_n, \beta_n|_k$, $k \geq 1$, are

$$\begin{aligned}
\text{(i)} \quad \epsilon_n &= 0 \left\{ \left(\frac{P_n R_{n-1} u_n}{p_n P_{n-1} U_n} \right) \left(\frac{\beta_n}{\alpha_n} \right)^{1-1/k} \right\}, \\
\text{(ii)} \quad \Delta \epsilon_n &= 0 \left\{ \left(\frac{u_n}{U_{n-1}} \right) \left(\frac{\beta_n}{\alpha_n} \right)^{1-1/k} \right\}.
\end{aligned}$$

PROOF. Sufficiency. Follows from Lemma 1.

Necessity of (i). Multiplying (2) by $\alpha_n^{1-1/k} p_n / P_n R_{n-1}$, the last term on the right becomes

$$\begin{aligned}
\frac{\alpha_n^{1-1/k} p_n}{P_n R_{n-1}} \tau_{n,5} &= - \frac{p_n P_{n-1} U_n}{P_n R_{n-1} u_n} \epsilon_n \Delta T_{n-1} \\
&= - \left\{ \frac{p_n P_{n-1} U_n}{P_n R_{n-1} u_n} \left(\frac{\alpha_n}{\beta_n} \right)^{1-1/k} \right\} \epsilon_n (\beta_n^{1-1/k} \Delta T_{n-1})
\end{aligned}$$

Following Bor [2]. By (3), it is possible to write the matrix transforming $(\beta_n^{1-1/k} \Delta T_{n-1})$ into $((\alpha_n^{1-1/k} p_n / P_n R_{n-1}) \tau_n)$. Since $|\overline{N}, p_n, \beta_n|_k$ implies $|N, R_n, \alpha_n|_k$, the matrix $\epsilon(l^k; l^k)$. By Lemma 3, a necessary condition for this implication is that the elements (in particular the diagonal elements) of this matrix should be bounded. Hence (i)

Necessity of (ii). Suppose $|\bar{N}, p_n, \beta_n|_k$ of $\sum a_n$ implies $|N, R_n, \alpha_n|_k$ of $\sum a_n \epsilon_n$. From (2)

$$|\tau_{n,4}| \leq \sum_r |\tau_{n,r}| + |\tau_n|, \quad r = 1, 2, 3, 5.$$

By Minkowski's inequality, using (i), we have, via the proof of Lemma 1,

$$\begin{aligned} \sum_{n=1}^m \alpha_n^{k-1} \left(\frac{p_n}{P_n R_{n-1}} \right)^k |\tau_{n,4}|^k &\leq 0(1) \left\{ \sum_{n=1}^m \alpha_n^{k-1} \left(\frac{p_n}{P_n R_{n-1}} \right)^k |\tau_{n,r}|^k + \sum_{n=1}^m \alpha_n^{k-1} \left(\frac{p_n}{P_n R_{n-1}} \right)^k |\tau_n|^k \right\} \\ &\leq 0(1) \sum_{n=1}^m |\Delta_n|^k \\ &= 0(1) \sum_{n=1}^m \beta_n^{k-1} |\Delta T_{n-1}|^k. \end{aligned}$$

Therefore

$$\sum_{n=1}^m \alpha_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \left| \frac{1}{R_{n-1}} \sum_{v=1}^{n-1} P_v q_{n-v-1} \frac{U_{v-1}}{u_v} \Delta \epsilon_v \Delta T_{v-1} \right|^k = 0(1) \sum_{n=1}^m \beta_n^{k-1} |\Delta T_{n-1}|^k$$

Now, put $\Delta T_{n-1} = \left(\frac{p_n}{P_n} \right) \left(\frac{\alpha_n}{\beta_n} \right)^{1-1/k}$, we obtain

$$\sum_{n=1}^m \alpha_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \left| \frac{1}{R_{n-1}} \sum_{v=1}^{n-1} p_v q_{n-v-1} \left\{ \left(\frac{U_{v-1}}{u_v} \right) \left(\frac{\alpha_v}{\beta_v} \right)^{1-1/k} \Delta \epsilon_v \right\} \right|^k = 0(1) \sum_{n=1}^m \alpha_n^{k-1} \left(\frac{p_n}{P_n} \right)^k$$

This should imply

$$\frac{1}{R_{n-1}} \sum_{v=1}^{n-1} p_v q_{n-v-1} \left\{ \left(\frac{U_{v-1}}{u_v} \right) \left(\frac{\alpha_v}{\beta_v} \right)^{1-1/k} \Delta \epsilon_v \right\} = 0(1).$$

But $\sum_{v=1}^{n-1} p_v q_{n-v-1} = R_{n-1}$, we get

$$\left(\frac{U_{v-1}}{u_v} \right) \left(\frac{\alpha_v}{\beta_v} \right)^{1-1/k} \Delta \epsilon_v = 0(1).$$

This completes the proof of the theorem.

REMARK. It is clear that

$$|\bar{N}, p_n, P_n/p_n|_k = |\bar{N}, p_n|_k, \quad |\bar{N}, p_n, n|_k = |R, p_n|_k, \quad |\bar{N}, 1, n|_k = |C, 1|_k,$$

and from our definition we may deduce that

$$q_n = 1 \Rightarrow |N, P_n, \phi_n|_k = |\bar{N}, p_n, \phi_n|_k,$$

which implies

$$|N, P_n, P_n/p_n|_k = |\bar{N}, p_n|_k$$

and

$$|N, P_n, n|_k = |R, p_n|_k \cdot p_n = 1 \Rightarrow |N, Q_n, n|_k = |N, q_n|_k.$$

4. APPLICATIONS

COROLLARY 1. Let $P_n u_n = 0(p_n U_n)$. Then the necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|\bar{N}, p_n|_k$ whenever $\sum a_n$ is summable $|\bar{N}, u_n|_k$, $k \geq 1$, are

$$\epsilon_n = 0 \left\{ \left(\frac{u_n P_n}{U_n p_n} \right)^{1/k} \right\}, \quad \Delta \epsilon_n = 0 \left\{ \left(\frac{u_n}{U_{n-1}} \right) \left(\frac{p_n}{P_n} \right)^{1-1/k} \right\}.$$

PROOF. Follows from the theorem by putting $q_n = 1$, $\alpha_n = P_n/p_n$, and $\beta_n = U_n/u_n$.

COROLLARY 2 (Bor and Thorpe [3]) Let $P_n u_n = 0(p_n U_n)$ and $p_n U_n = 0(P_n u_n)$. Then $\sum a_n$ is summable $|\bar{N}, p_n|_k$ iff it is summable $|\bar{N}, u_n|_k$, $k \geq 1$.

PROOF. Follows from Corollary 1 by putting $\epsilon_n = 1$.

COROLLARY 3. Let $Q_{n-1} u_n = 0(U_n)$. Then the necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|N, q_n|_k$ whenever $\sum a_n$ is summable $|\bar{N}, u_n|_k$, $k \geq 1$, are

$$\epsilon_n = 0 \left\{ \left(\frac{Q_{n-1}}{n} \right) \left(\frac{n u_n}{U_n} \right)^{1/k} \right\}, \quad \Delta \epsilon_n = 0 \left\{ \left(\frac{u_n}{U_{n-1}} \right) \left(\frac{U_n}{n u_n} \right)^{1-1/k} \right\}.$$

PROOF. Follows from the theorem by putting $p_n = 1$, $\alpha_n = n$, $\beta_n = U_n/u_n$ and making use of Lemma 2.

COROLLARY 4. Let $Q_{n-1} u_n = 0(U_n)$. Then a necessary and sufficient condition that $\sum a_n$ be summable $|N, q_n|_k$ whenever it is summable $|\bar{N}, u_n|_k$, $k \geq 1$, is

$$n^{k-1} U_n = 0(Q_{n-1}^k u_n).$$

PROOF. Follows from Corollary 3 by putting $\epsilon_n = 1$.

COROLLARY 5. Let $\{n^{1-1/k} p_n / P_n P_{n-1}\}$ nonincreasing, $P_n u_n = 0(p_n U_n)$, and

$$\sum_{n=v}^{\infty} \frac{n^{k-1} p_n^k}{P_n^k P_{n-1}} = 0 \left(\frac{v^{k-1} p_v^{k-1}}{P_v^k} \right).$$

Then the necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|R, p_n|_k$ whenever $\sum a_n$ is summable $|\bar{N}, u_n|_k$, $k \geq 1$, are

$$\epsilon_n = 0 \left\{ \left(\frac{P_n}{n p_n} \right) \left(\frac{n u_n}{U_n} \right)^{1/k} \right\}, \quad \Delta \epsilon_n = 0 \left\{ \left(\frac{u_n}{U_{n-1}} \right) \left(\frac{U_n}{n u_n} \right)^{1-1/k} \right\}.$$

PROOF. Follows from the theorem by putting $q_n = 1$, $\alpha_n = n$ and $\beta_n = U_n/u_n$.

COROLLARY 6. Let $P_n u_n = 0(p_n U_n)$. Then the necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|\bar{N}, p_n|_k$ whenever $\sum a_n$ is summable $|R, u_n|_k$, $k \geq 1$, are

$$\epsilon_n = 0 \left\{ \left(\frac{n u_n}{U_n} \right) \left(\frac{P_n}{n p_n} \right)^{1/k} \right\}, \quad \Delta \epsilon_n = 0 \left\{ \left(\frac{u_n}{U_{n-1}} \right) \left(\frac{n p_n}{P_n} \right)^{1-1/k} \right\}.$$

PROOF. Follows from the theorem by putting $q_n = 1$, $\alpha_n = P_n/p_n$ and $\beta_n = n$.

The following four results follows from Corollary 3 and they are generalizations for the results of [8].

COROLLARY 7. The necessary and sufficient conditions that $\sum a_n \epsilon_n$ are summable $|C, \alpha|_k$, $0 \leq \alpha \leq 1$, whenever $\sum a_n$ is summable $|C, 1|_k$, $k \geq 1$, are

$$\epsilon_n = 0(n^{\alpha-1}), \quad \Delta \epsilon_n = 0(n^{-1}).$$

PROOF. Follows by putting $q_n = A_n^{\alpha-1}$, $u_n = 1$.

COROLLARY 8. The necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|N, 1/(n+1)|_k$ whenever $\sum a_n$ is summable $|C, 1|_k$, $k \geq 1$, are

$$\epsilon_n = 0(\log n/n), \quad \Delta \epsilon_n = 0(n^{-1}).$$

PROOF. Follows by putting $q_n = 1/(n+1)$, $u_n = 1$

COROLLARY 9. The necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|N, 1/(n+1)|_k$ whenever $\sum a_n$ is summable $|R, \log n, 1|_k$, $k \geq 1$, are

$$\epsilon_n = O\left\{(\log n)^{1-1/k}/n\right\}, \quad \Delta\epsilon_n = O\left\{1/n(\log n)^{1/k}\right\}.$$

PROOF. Follows by putting $q_n = u_n = 1/(n+1)$.

COROLLARY 10. The necessary and sufficient conditions that $\sum a_n \epsilon_n$ be summable $|C, \alpha|_k$, $0 \leq \alpha \leq 1$, whenever $\sum a_n$ is summable $|R, \log n, 1|_k$, $k \geq 1$, are

$$\epsilon_n = O\left\{n^{\alpha-1}/(\log n)^{1/k}\right\}, \quad \Delta\epsilon_n = O\left\{1/n(\log n)^{1/k}\right\}.$$

PROOF. Follows by putting $q_n = A_n^{\alpha-1}$, $u_n = 1/(n+1)$.

Lastly it may be mentioned that many other results could be obtained either from the theorem or from its corollaries.

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