ABSTRACT. Let $X$ be a completely regular Hausdorff space, $E$ a topological vector space, $V$ a Nachbin family of weights on $X$, and $CV_0(X, E)$ the weighted space of continuous $E$-valued functions on $X$. Let $\theta : X \to C$ be a mapping, $f \in CV_0(X, E)$ and define $M_\theta(f) = \theta f$ (pointwise). In case $E$ is a topological algebra, $\psi : X \to E$ is a mapping then define $M_\psi(f) = \psi f$ (pointwise). The main purpose of this paper is to give necessary and sufficient conditions for $M_\theta$ and $M_\psi$ to be the multiplication operators on $CV_0(X, E)$ where $E$ is a general topological space (or a suitable topological algebra) which is not necessarily locally convex. These results generalize recent work of Singh and Manhas based on the assumption that $E$ is locally convex.

KEY WORDS AND PHRASES: Nachbin family of weights, topological vector spaces, vector-valued continuous functions, weighted topology, multiplication operators, locally idempotent topological algebras.


1. INTRODUCTION

The fundamental work on weighted spaces of continuous scalar-valued functions has been done mainly by Nachbin [9,10] in the 1960's. Since then it has been studied extensively for a variety of problems such as weighted approximation, characterization of the dual space, approximation property, description of inductive limit and of tensor-product, etc for both scalar- and vector-valued functions (for instance see [1-5,8-14]). Recently Singh and Summers [13] have studied the notion of composition operators on $CV_0(X, C)$. Later, Singh and Manhas [12] made an analogous study of multiplication operators on $CV_0(X, E)$, assuming $E$ to be a locally convex space or a locally $m$-convex algebra. The purpose of this paper is to generalize the results of Singh and Manhas [12] to the case when $E$ is a general topological vector space which is not necessarily locally convex. Section 3 contains our main results while section 2 is devoted to some technical preliminaries required for the development of our results.

2. PRELIMINARIES

Throughout this paper we shall assume, unless stated otherwise, that $X$ is a completely regular Hausdorff space and $E$ is a non-trivial Hausdorff topological vector space. Let $S^+(X)$ denote the set of...
all non-negative upper-semicontinuous functions on $X$, and let $S^+_\infty(X)$ (respectively $S^-\infty(X)$), be the subset of $S^+(X)$ consisting of those functions vanishing at infinity (respectively having compact support).

A *Nachbin family* on $X$ is a subset $V$ of $S^+_\infty(X)$ such that, given $u, v \in V$, there exist $w \in V$ and $t > 0$ so that $u, v \leq tw$ (pointwise); the elements of $V$ are called *weights*. Let $C(X, E)$ ($C_b(X, E)$) be the vector space of all continuous (and bounded) $E$-valued functions on $X$, and let $CV_b(X, E)$ ($CV_0(X, E)$) denote the subspace of $C(X, E)$ consisting of those $f$ such that $vf$ is bounded (vanishes at infinity) for each $v \in V$. When $E = C$ (or $R$), these spaces are denoted by $C(X), C_b(X), CV_b(X),$ and $CV_0(X)$.

If $\phi \in C(X)$ and $a \in E$, then $\phi \otimes a$ is a function in $C(X, E)$ defined by $(\phi \otimes a)(x) = \phi(x)a(x) \in X)$. If $U$ and $V$ are two Nachbin families on $X$ and, for each $u \in U$, there is a $v \in V$ such that $u \leq v$, then we write $U \leq V$. If, for each $x \in X$, there is a $v \in V$ with $v(x) \neq 0$, we write $V > 0$. For any function $\theta : X \to C$, we let $V[\theta] = \{v[\theta] : v \in V\}$.

Given any Nachbin family $V$ on $X$, the *weighted topology* $\omega_\nu$ on $CV_0(X, E)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$N(v, G) = \{f \in CV_0(X, E) : (vf)(X) \subseteq G\},$$

where $v \in V$ and $G$ is a neighborhood of 0 in $E$; $CV_0(X, E)$ endowed with $\omega_\nu$ is called a *weighted space*. We mention that if $V = S^+_\infty(X)$, then $CV_b(X, E) = CV_0(X, E) = C_b(X, E)$ and $\omega_\nu = \beta$, the strict topology and write as $(C_b(X, E), \beta)$; if $V = S^-\infty(X)$, then $CV_0(X, E) = CV_0(X, E) = C(X, E)$ and $\omega_\nu = k$, the compact-open topology and we write as $(C(X, E), k)$. For more information on weighted spaces, we refer to [1-2,9-14] when $E$ is a scalar field or a locally convex space and to [1,3-5,8] in the general setting.

Let $\theta : X \to C$ and $\psi : X \to E$ be two mappings, and let $L(X, E)$ be the vector space of all functions from $X$ into $E$. The scalar multiplication on $E$ and, in case $E$ is an algebra, multiplication on $E$ give rise to two linear mappings $M_\theta$ and $M_\psi$ from $CV_b(X, E)$ into $L(X, E)$ defined by $M_\theta(f) = \theta f$ and $M_\psi(f) = \psi f$, where the product of functions is defined pointwise. If $M_\theta$ and $M_\psi$ map $CV_b(X, E)(CV_0(X, E))$ into itself and are continuous, they are called *multiplication operators* on $CV_b(X, E)(CV_0(X, E))$ induced by $\theta$ and $\psi$, respectively.

A neighborhood $G$ of 0 in $E$ is called *shrinkable* if $rG \subseteq \text{int} G$ for $0 \leq r < 1$. By ([6], Theorems 4 and 5), every Hausdorff topological vector space has a base of shrinkable neighborhoods of 0 and also the Minkowski functional $\rho_G$ of any such neighborhood $G$ is continuous.

Now let $E$ be a topological algebra with jointly continuous multiplication and having $W$, a base of neighborhoods of 0. Then, given any $G \in W$, there exists an $H \in W$ such that $H^2 \subseteq G$. (Here $H^2 = \{ab : a, b \in H\}$.) A subset $G \in W$ is called *idempotent* (or multiplicative) if $G^2 \subseteq G$. Following Zelazko ([16], p. 31), $E$ is said to be a *locally idempotent algebra* if it has a base of neighborhoods of 0 consisting of idempotent sets. It is easily seen that if $G \in W$ is idempotent, then $\rho_G$ is submultiplicative: $\rho_G(ab) \leq \rho_G(a)\rho_G(b)$ for all $a, b \in E$; further, if $E$ has an identity $e$, $\rho_G(e) \geq 1$. The notion of locally idempotent algebras is a strict generalization of the notion of locally $m$-convex algebras introduced by Michael [7] (see also [15, p. 348]).

### 3. Characterization of Multiplication Operators

In this section, we give necessary and sufficient conditions for $M_\theta$ and $M_\psi$ to be the multiplication operators on the weighted space $CV_0(X, E)$. (These results hold also for the space $CV_b(X, E)$ with slight modification in the proofs and are therefore omitted.) To avoid trivial cases we assume that the Nachbin family $V$ on $X$ satisfies the following conditions

(*) $V > 0$,

(**) corresponding to each $x \in X$, there exists an $h_x \in CV_0(X)$ such that $h_x(x) \neq 0$. (This holds in particular, when each $v$ in $V$ vanishes at infinity or $X$ is locally compact.)

**Theorem 3.1.** For a mapping $\theta : X \to C$, the following are equivalent:
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(a) $\theta$ is continuous and $V |\theta| \leq V$;
(b) $M_\theta$ is a multiplication operator on $CV_0(X, E)$.

**Proof.** Let $W$ be a base of closed, balanced, and shrinkable neighborhoods of 0 in $E$.

(a) $\Rightarrow$ (b). We first show that $M_\theta$ maps $CV_0(X, E)$ into itself. Let $f \in CV_0(X, E)$, and let $v \in V$ and $G \in W$. Choose $u \in V$ such that $\|\theta\| \leq u$. There exists a compact set $K \subseteq X$ such that $u(x)f(x) \in G$ for all $x \in X \setminus K$. Then, since $G$ is balanced,

$$v(x)M_\theta(f)(x) = v(x)\theta(x)f(x) \in G$$

for all $x \in X \setminus K$. Hence $vM_\theta(f)$ vanishes at infinity; further, since $\theta$ is continuous, $M_\theta(f) \in CV_0(X, E)$. To prove the continuity of $M_\theta$, let $\{f_\alpha\}$ be a net in $CV_0(X, E)$ with $f_\alpha \to 0$. Let $v, G$ and $u$ be chosen as above. Choose an index $\alpha_0$ such that $f_\alpha \in N(u, G)$ for all $\alpha \geq \alpha_0$. Then it follows that $\theta f_\alpha \in N(v, G)$ for all $\alpha \geq \alpha_0$. Thus $M_\theta(f_\alpha) \to 0$. So $M_\theta$ is continuous at 0 and hence, by linearity, it is continuous on $CV_0(X, E)$.

(b) $\Rightarrow$ (a). We first show that $\theta$ is continuous. Let $\{x_\alpha\}$ be a net in $X$ with $x_\alpha \to x \in X$. By assumption (**), there exists an $h \in CV_0(X)$ such that $h(x) \neq 0$. Since $M_\theta$ is a self-map on $CV_0(X, E)$, it follows that the function $\theta h$ from $X$ into $C$ is continuous. Hence $\theta(x_\alpha)h(x_\alpha) \to \theta(x)h(x)$ and consequently $\theta(x_\alpha) \to \theta(x)$. We next show that $V |\theta| \leq V$. Let $v \in V$.

By continuity of $M_\theta$, given $G \in W$, there exist $u \in V$ and $H \in W$ such that

$$M_\theta(N(u, H)) \subseteq (v, G).$$

Without loss of generality we may assume that $G \cup H$ is a proper subset of $E$. Choose $a \in X \setminus (G \cup H)$, and put $t = \rho_H(a)/\rho_G(a)$. We claim that $v|\theta| \leq 2tu$. Fix $x_0 \in X$. We shall consider two cases $u(x_0) \neq 0$ and $u(x_0) = 0$.

Suppose that $u(x_0) \neq 0$, and let $\epsilon = u(x_0)$. Then $D = \{x \in X : u(x) < 2\epsilon\}$ is an open neighborhood of $x_0$. Using the complete regularity of $X$ and the assumption (**), there is an $h \in CV_0(X, E)$ with $0 \leq h \leq 1$, $h(x_0) = 1$, and $h(X \setminus D) = 0$. Define $f = (h \otimes a)/2t \rho_H(a)$. Since $\rho_H$ is homogeneous, for any $x \in X$,

$$\rho_H(u(x)f(x)) = u(x)h(x)/2\epsilon < 1,$$

by considering the cases $x \in D$ and $x \in X \setminus D$. Since $H = \{b \in E : \rho_H(b) \leq 1\}$, we have $f \in N(u, H)$. Hence, by (1), $\theta f \in N(v, G)$. This implies that, for any $x \in X$,

$$\rho_G(\theta(x)u(x)h(x)a/2t \rho_H(a)) \leq 1,$$

or $v(x)\theta(x)|\theta(x)| \leq 2tu(x_0)$. In particular, $v(x_0)|\theta(x_0)| \leq 2tu(x_0)$.

Now suppose that $u(x_0) = 0$ but $v(x_0)|\theta(x_0)| > 0$. Put $\epsilon = v(x_0)|\theta(x_0)|/2t$. Let $D = \{x \in X : u(x) < \epsilon\}$, and choose an $h \in CV_0(X)$ as above. Define $g = (h \otimes a)/\epsilon \rho_H(a)$. We easily have $g \in N(u, H)$, and hence $\theta g \in N(v, G)$. From this we obtain

$$v(x_0)|\theta(x_0)| \leq te = v(x_0)|\theta(x_0)|/2t,$$

which is impossible unless $v(x_0)|\theta(x_0)| = 0$. This completes the proof.

We next consider the case of the operator $M_\varphi$.

**Theorem 3.2.** Let $E$ be a Hausdorff locally idempotent algebra with identity $e$ and $W$ a base of neighborhoods of 0. Then, for a mapping $\psi : X \to E$, the following are equivalent:
(a) $\psi$ is continuous and $V \rho_G \circ \psi \leq V$ for every $G \in W$.
(b) $M_\psi$ is a multiplication operator on $CV_0(X, E)$.

**Proof.** We may assume that $W$ consists of closed, balanced, shrinkable, and idempotent sets.
(a) $\Rightarrow$ (b) We first show that $M_\psi$ maps $CV_0(X, E)$ into itself Let $f \in CV_0(X, E)$, and let $v \in V$ and $G \in W$. Choose $u \in V$ such that $V \rho_G \circ \psi \leq u$. There exists a compact set $K \subseteq X$ such that $u(x)f(x) \in G$ for all $x \in X \setminus K$. Since $\rho_G$ is submultiplicative, for any $x \in X \setminus K$, we have

$$\rho_G(u(x)f(x)) \leq u(x)\rho_G(f(x)) \leq u(x)\rho_G(f(x)) \leq 1;$$

hence $M_\psi(f) \in CV_0(X, E)$. Using again the submultiplicativity of $\rho_G$, the continuity of $M_\psi$ follows in the same way as in the proof of Theorem 1.

(b) $\Rightarrow$ (a). Let $\{x_\alpha\}$ be a net in $X$ such that $x_\alpha \to x \in X$. Choose an $h \in CV_0(X)$ with $h(x) \neq 0$. Since $M_\psi$ is a self-map on $CV_0(X, E)$, it follows that the function $\psi(h \otimes a)$ from $X$ into $E$ is continuous. Hence $h(x_\alpha)\psi(x_\alpha) \to h(x)\psi(x)$ and consequently $\psi(x_\alpha) \to \psi(x)$ This proves the continuity of $\psi$. Next, let $v \in V$ and $G \in W$. There exist $u \in V$ and $H \in W$ such that

$$M_\psi(N(u, H)) \subseteq N(v, G).$$

Without loss of generality, we may assume that $H$ is a proper subset of $E$. We claim that

$$\nu \rho_G \circ \psi \leq 2\rho_H(e)u.$$  

Fix $x_0 \in X$. First assume that $u(x_0) \neq 0$, and let $\epsilon = u(x_0)$. Then $D = \{x \in X : u(x) < 2\epsilon\}$ is an open neighborhood of $x_0$, so there exists an $h \in CV_0(X)$ such that $0 \leq h \leq 1, h(x_0) = 1$, and $h(X \setminus D) = 0$. Define $f = (h \otimes e)/2\rho_H(e)$. Then, for any $x \in X$,

$$\rho_H(u(x)f(x)) = \rho_H(u(x)h(x)e)/2\rho_H(e) \leq 1;$$

that is, $f \in N(u, H)$. Hence, by (2), $\psi f \in N(v, G)$. This implies that, for any $x \in X$,

$$v(x)h(x)\rho_G(\psi(x)) \leq 2\rho_H(e).$$

In particular, $v(x_0)\rho_G(\psi(x_0)) \leq 2\rho_H(e)u(x_0)$. Next suppose that $u(x_0) = 0$, but $v(x_0)\rho_G(\psi(x_0)) > 0$. Put $\epsilon = v(x_0)\rho_G(\psi(x_0))/2\rho_H(e)$. Let $D = \{x \in X : u(x) < \epsilon\}$, and choose an $h \in CV_0(X)$ as above. Define $g = (h \otimes e)/\epsilon\rho_H(e)$. Then $g \in N(u, H)$, so by (2), $\psi g \in N(v, g)$. From this we obtain

$$v(x_0)\rho_G(\psi(x_0)) \leq \rho_H(e) \epsilon = v(x_0)\rho_G(\psi(x_0))/2,$$

which is impossible unless $v(x_0)\rho_G(\psi(x_0)) = 0$. This completes the proof.

Finally, we apply the above results to the cases: $V = S^+_\epsilon(X)$ and $V = S^+_0(X)$ and obtain the following.

**THEOREM 3.3.**

(i) If $0 \to X \to C$ is a continuous mapping, then $M_\theta$ is a multiplication operator on $(C(X, E), k)$.

(ii) If $E$ is a Hausdorff locally idempotent algebra with identity $e$ and $\psi : X \to E$ a continuous mapping, then $M_\psi$ is a multiplication operator on $(C(X, E), k)$.

**PROOF.** (i) In view of Theorem 1, we only need to verify that $V|\theta| \leq V$, where $V = S^+_\epsilon(X)$. Let $v \in V$. Choose a compact set $K \subseteq X$ with $v(x) = 0$ for all $x \in X \setminus K$. Let $s = \sup\{|\theta(x)| : x \in K\}$ and $t = \sup\{v(x) : x \in K\}$, and let $u = st_XK$. Then $u \in V$ and clearly $v(x)|\theta(x)| \leq u(x)$ for all $x \in X$.

(ii) Let $W$ be a base of neighborhoods of 0 in $E$ consisting of closed, balanced, shrinkable, and idempotent sets. In view of Theorem 2, we only need to verify that $V \rho_G \circ \psi \leq V$ for every $G \in W$, where $V = S^+_\epsilon(X)$. Let $v \in V$ and $G \in W$. Choose a compact set $K \subseteq X$ with $v(x) = 0$ for all $x \in X \setminus K$. Let $s = \sup\{\rho_G(\psi(x)) : x \in K\}$ and $t = \sup\{v(x) : x \in K\}$, and let $u = st_XK$. Then $u \in V$ and clearly $v(x)\rho_G(\psi(x)) \leq u(x)$ for all $x \in X$. This completes the proof of the theorem.

**REMARK.** The above result need not hold for the space $(C_0(X, E), \beta)$. To see this, consider $X = R^+$, $E = C$, and $V = S^+_0(X)$. Let $\theta = \psi : X \to C$ be a mapping given by $\theta(x) = x^2(x \in X)$, and let $v \in V$ be given by $v(x) = \frac{1}{x}(x \in X)$. Then $v(x)|\theta(x)| = x$ for all $x \in X$. Since each $u \in V$ is a bounded function, $v|\theta| \leq u$ for every $u \in V$. Hence $V|\theta| \leq V$ does not hold and so, by Theorem 1, $M_\theta$
is not a multiplication operator on \((C_b(X), \beta)\). The same is also true for the space \((C_b(X), u)\), where \(u\) is the uniform topology, since \(\beta \leq u\). However, if \(\theta\) and \(\psi\) are bounded continuous functions, then it is easily seen that \(M_\theta\) and \(M_\psi\) are always multiplication operators on \(CV_0(X, E)\) for any Nachbin family \(V\).

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