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A SCHUR METHOD FOR THE SOLUTION OF THE MATRIX RICCATI EQUATION

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ABSTRACT. This paper is concerned with an analytic solution of the finite-time matrix Riccati equation. The solution to the Riccati equation is given in terms of multiple of two matrices. These matrices are found using a Schur-type decomposition for Hamiltonian matrices. Simple examples illustrating the method are presented.

KEYWORRDS AND PHRASES. Schur method, matrix Riccati, analytic solution.
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1. INTRODUCTION.

The matrix Riccati equation appears in most optimal control systems design problems. This equation, in one form or another, has an important role in linear state and output regulator tracking, multi-variable and large scale systems, scattering theory and estimation [1,2].

It is known that the boundedness of the solution of the matrix Riccati equation is equivalent to the "no-conjugate point to the final time" [3-5]. The solution of this equation is difficult to obtain from two points of view. One is that it is nonlinear, and the other being in matrix form. In the present paper the solution to the matrix Riccati equation is expressed in terms of multiple of two matrices. These matrices are then obtained by using a Schur-type decomposition for Hamiltonian matrices. Illustrative examples are presented to show the feasibility of this method.

2. STATEMENT OF THE PROBLEM

Consider the linear time invariant system,

$$\dot{x}(t) = Ax(t) + Bu(t), x(t_0) = x_0 \tag{2.1}$$

where $x(t)$ and $u(t)$ are $n \times 1$ and $m \times 1$ state and control vectors, respectively. A and B are constant $n \times n$ and $n \times m$ matrices. The optimal control problem would be to find an optimal control $u(t)$ satisfying (2.1) while minimizing the quadratic cost functional,

$$J = \frac{1}{2}x^T(t_f)Sx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt \tag{2.2}$$

where t_0 and t_f are the initial and final times which are given, S, Q, and R are constant matrices of appropriate dimensions, R is symmetric positive definite, while S and Q are symmetric positive semi-definite, and T denotes transpositions. The optimal control is given by [6]

$$u(t) = -R^{-1}B^TW(t)x(t)$$

where $W(t)$ is the solution of the matrix Riccati equation

$$\dot{W} = -WD_1 - D_1^T W - WD_2 W + D_3,\ W(t_f) = S \tag{2.3}$$

with

$$D_1 = A,\ D_2 = -BR^{-1}B^T,\ D_3 = -Q. \tag{2.4}$$

Usually, the infinite time problem is considered, in this case $\dot{W} = 0$ and the Riccati equation is reduced to an algebraic equation. The solution of the matrix Riccati equation has been considered in [6-8]. The solution of (2.3) can be expressed as [8].

$$W(t) = Y(t)X^{-1}(t) \text{ with } Y(t_f) = S,\ X(t_f) = I$$

where

$$\dot{Z} = DZ \tag{2.5}$$

with

$$Z(t) = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \text{ and } D = \begin{bmatrix} D_1 & D_2 \\ D_3 & -D_1^T \end{bmatrix} \tag{2.6}$$

The solution of (2.5) is given by

$$Z(t) = e^{D(t-t_f)} \begin{bmatrix} X(t_f) \\ Y(t_f) \end{bmatrix} \tag{2.7}$$

Let $t - t_f = t_1$. Then (2.7) can be written as

$$Z(t) = M \exp\,(M^{-1}DMt_1)M^{-1} \begin{bmatrix} I \\ S \end{bmatrix} \tag{2.8}$$

where M is a $2n \times 2n$ nonsingular matrix. It is known that [9]

$$e^{\begin{bmatrix} A & B \\ O & C \end{bmatrix} t} = \begin{bmatrix} e^{At} & \int_0^t e^{A(t-s)} B e^{Cs} ds \\ 0 & e^{Ct} \end{bmatrix} \tag{2.9}$$

where A, B and C are square matrices. To find $Z(t)$ in (2.8) we find M which makes $M^{-1}DM$ a quasi-upper-triangular. This can be achieved using a Schur method and leads to a method for obtaining the solution to the Riccati equation.

3. SOLUTION OF THE MATRIX RICCATI EQUATION USING A SCHUR METHOD

We will use the following theorems.

THEOREM 1. (Schur canonical form). There exists a unitary matrix U such that $U^H GU$ is triangular and has as diagonal elements the eigenvalues of G. Also there exists an orthogonal matrix P such that $P^T GP$ is quasi- upper-triangular where each diagonal element is either a 1×1 matrix or a 2×2 matrix having complex conjugate eigenvalues [10].

THEOREM 2. (The real Schur-Hamilton decomposition). If D has no pure imaginary eigenvalues, then there exist an orthogonal matrix Q

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix} \quad Q_{11}, Q_{12} \epsilon R^{n \times n} \tag{3.1}$$

such that

$$G = Q^T DQ = \begin{bmatrix} E & F \\ 0 & -E^T \end{bmatrix} \quad E, F \epsilon R^{n \times n} \tag{3.2}$$

where E is upper quasitriangular with $F^T = F$. Also, Q can be chosen such that the eigenvalues of E are in the left half plane and moreover, each 2×2 diagonal block of E appears with a complex conjugate pair of eigenvalues [11].

If D has no pure imaginary eigenvalues, then we choose M in (2.8) using Theorem 2. Hence, we have

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix} \begin{bmatrix} e^{Et_1} & \int_0^{t_1} e^{E(t_1-s)} F e^{-E^T s} ds \\ 0 & e^{-Et_1} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix}^T \begin{bmatrix} I \\ S \end{bmatrix} \tag{3.3}$$

and if D has pure imaginary eigenvalues, we use Theorem 1. Consequently, matrices X and Y can be calculated. The results obtained are illustrated by the following examples.

4. EXAMPLES

Example 1. Consider minimizing the cost functional

$$J = \frac{1}{2}\int_0^{t_f} \left[x^T(t) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x(t) + u^T(t)u(t) \right] dt, \tag{4.1}$$

subject to the conditions

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),\ x(0) = x_0. \tag{4.2}$$

In this problem using (2.6), we obtain

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \end{bmatrix}. \tag{4.3}$$

The eigenvalues of D are $-1, -1, 1$ and 1. The matrix Q,

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{5}}{10} & -\frac{1}{2} & -\frac{3\sqrt{5}}{10} \\ -\frac{1}{2} & \frac{\sqrt{5}}{10} & \frac{1}{2} & -\frac{3\sqrt{5}}{10} \\ \frac{1}{2} & \frac{3\sqrt{5}}{10} & \frac{1}{2} & \frac{\sqrt{5}}{10} \\ -\frac{1}{2} & \frac{3\sqrt{5}}{10} & -\frac{1}{2} & \frac{\sqrt{5}}{10} \end{bmatrix} \tag{4.4}$$

in an orthogonal matrix which redures D to

$$G = Q^T D Q = \begin{bmatrix} E & F \\ 0 & -E^T \end{bmatrix} \tag{4.5}$$

with

$$E = \begin{bmatrix} -1 & \frac{2\sqrt{5}}{5} \\ 0 & -1 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & -\frac{\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} & 1 \end{bmatrix} \tag{4.6}$$

also, e^{Et_1} and $e^{-E^T t_1}$ can be calculated as

$$e^{Et_1} = \begin{bmatrix} e^{-t_1} & \frac{2\sqrt{5}}{5} t_1 e^{-t_1} \\ 0 & e^{-t_1} \end{bmatrix} \text{ and } e^{-E^T t_1} = \begin{bmatrix} e^{t_1} & 0 \\ -\frac{2\sqrt{5}}{5} t_1 e^{t_1} & e^{t_1} \end{bmatrix}. \tag{4.7}$$

Hence, using (3.3) with $S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, we obtain

$$X = \frac{1}{2} \begin{bmatrix} t_1 \cosh t_1 - t_1 \sinh t_1 & \sinh t_1 + t_1 \cosh t_1 \\ \sinh t_1 - t_1 \cosh t_1 & 2\cosh t_1 + t_1 \sinh t_1 \end{bmatrix} \tag{4.8}$$

$$Y = \frac{1}{2} \begin{bmatrix} t_1 \cosh t_1 - 3\sinh t_1 & -t_1 \sinh t_1 \\ t_1 \sinh t_1 & -(3\sinh t_1 + t_1 \cosh t_1) \end{bmatrix}. \tag{4.9}$$

Then

$$W = YX^{-1}$$
$$\frac{1}{3\cosh 2t_1 + 2t_1^2 + 5} \begin{bmatrix} -6\sinh 2t_1 + 4t_1 & 3\cosh 2t_1 - 2t_1^2 - 3 \\ 3\cosh 2t_1 - 2t_1^2 - 3 & -6\sinh 2t_1 + 4t_1 \end{bmatrix} \tag{4.10}$$

Example 2. The following example was first given in [9].

Find the solution of the following Riccati equation:

$$\dot{P} = P^2 + 2P + 2, \; P(t_f) = 1. \tag{4.11}$$

For this problem, the matrix D is given by

$$D = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}, \tag{4.12}$$

and the matrix

$$U = \frac{1}{\sqrt{6}} \begin{bmatrix} i-1 & 2 \\ 2 & i+1 \end{bmatrix} \tag{4.13}$$

is a unitary matrix which reduces D to

$$U^H D U = \begin{bmatrix} i & i+2 \\ 0 & -i \end{bmatrix}. \tag{4.14}$$

Then

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{6} \begin{bmatrix} i-1 & 2 \\ 2 & i+1 \end{bmatrix} \begin{bmatrix} e^{it_1} & (i+2)\sin t_1 \\ 0 & e^{-it_1} \end{bmatrix} \begin{bmatrix} -i-1 & 2 \\ 2 & -i+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{4.15}$$

and

$$P(t) = \frac{Y}{X} = \frac{\cos t_1 + 3\sin t_1}{\cos t_1 - 2\sin t_1}.$$

This is in agreement with the result obtained in [9].

5. CONCLUSION

A method is proposed for the solution of the matrix Riccati equation which has several applications in optimal control problems. This method transforms the problem into examining the exponential of the Hamiltonian matrix. This exponential is found using a Schur decomposition for Hamiltonian matrices. Examples illustrating the concept involved are included.

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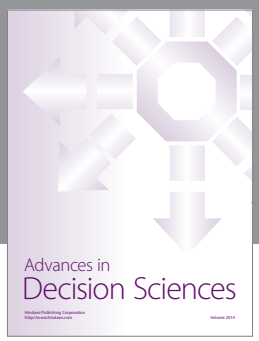

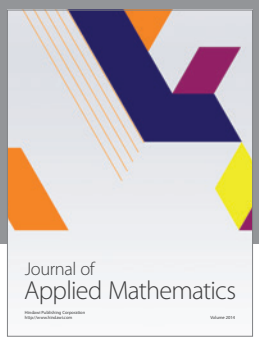

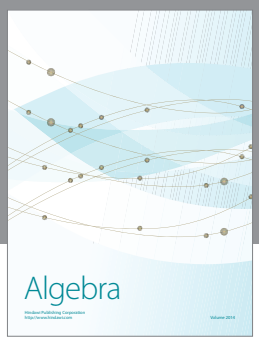

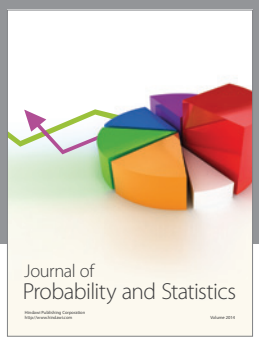

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