A NEW LOOK AT MEANS ON TOPOLOGICAL SPACES

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ABSTRACT. We use methods of algebraic topology to study when a connected topological space admits an n-mean map.

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1. INTRODUCTION

Carathéodory and Aumann (see [1],[2]) were among the pioneers who first considered the question of what path-connected regions X in \( \mathbb{R}^n \) or \( \mathbb{C}^n \) could support an n-mean, that is, a map \( \mu : X^n \to X \) satisfying

(i) \( \mu \circ \Delta = 1; \) \( X \to X \), where \( \Delta \) is the diagonal map \( \Delta : X \to X^n \); and

(ii) \( \mu \sigma = \mu : X^n \to X \), where \( \sigma \in S_n \), the symmetric group on n letters, acting on \( X^n \) by permuting components. One of their main concerns was to find out if the existence of such an n-mean, \( n \geq 2 \), implied that X was simply connected.

In 1954, Beno Eckmann [4] attacked the question with the tools of algebraic topology. He supposed X to be a polyhedron and only required conditions (i), (ii) above up to homotopy. One of his principal conclusions was that if X is compact and admits a (homotopy) n-mean for all n, then X is contractible.

In 1962, Eckmann, together with Tudor Ganea and the author, returned to the study of n-means in a more general setting (see [5]). Thus the n-mean defined in [4] was a morphism in the category \( T_h \) of based connected CW-complexes and based homotopy classes of based maps. In this generality one was able to exploit the idea of mean-preserving functors. Thus if \( C, D \) are categories with products and \( F : C \to D \) is a product-preserving functor, then \( F \mu \) is an \( n \)-mean in \( D \) for any \( n \)-mean \( \mu \) in \( C \). Moreover, one could also examine the dual question of the existence of \( n \)-comeans.

It turns out that the concept of \( P \)-local objects and \( P \)-localization, where \( P \) is a family of primes, and the results related to these concepts in the categories \( T_h \) and \( N \), the category of nilpotent groups (see [6]), enable one to simplify many arguments in [5] and to extend the results of that paper.

2. MEANS IN THE CATEGORY OF GROUPS

Let \( \mathcal{G} \) be the category of groups. Let \( n \) be an integer, \( n \geq 2 \), and let \( P \) be the family of primes \( p \) such that \( p \mid n \). We then prove

**Theorem 2.1.** The group \( G \) admits an \( n \)-mean \( \mu \) in \( \mathcal{G} \) if and only if \( G \) is commutative and \( P \)-local. In that case, if we write \( G \) additively, \( \mu \) is given by

\[
\mu(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}.
\]
\[ \mu(g_1, g_2, \ldots, g_n) = \frac{1}{n}(g_1 + g_2 + \cdots + g_n). \] (2.1)

**PROOF.** Note first that if \( G \) is commutative, then \( G \) is \( P \)-local if and only if \( G \) admits unique division by \( n \). It is then plain that (2.1) defines an \( n \)-mean on \( G \).

Conversely, let \( \mu \) be an \( n \)-mean on \( G \). For \( g, h \in G \) (at this stage, we write \( G \) multiplicatively), set 
\[ \mu(g, e, \ldots, e) = \gamma, \mu(h, e, \ldots, e) = \delta. \] Then, by condition (ii),
\[ \mu(e, g, \ldots, e) = \cdots = \mu(e, e, \ldots, g) = \gamma, \]
so that, by condition (i),
\[ g = \mu(g, g, \ldots, g) = \gamma^n. \]
Similarly, \( h = \delta^n \). But \( \mu(g, h, \ldots, e) = \gamma \delta, \mu(h, g, \ldots, e) = \delta \gamma, \) and \( \mu(g, h, \ldots, e) = \mu(h, g, \ldots, e) \). Thus \( \gamma \) commutes with \( \delta \), so that \( g \) commutes with \( h \) and \( G \) is commutative. To show that \( G \) is \( P \)-local it remains to show that \( n^{th} \) roots are unique in \( G \). But, again using properties (i) and (ii), we conclude that \( \mu(g^n, e, \ldots, e) = \mu(g, g, \ldots, g) = g \), so that \( g \) is determined by \( g^n \). Thus \( G \) is commutative and \( P \)-local and, writing additively, we have
\[ \mu(g_1, g_2, \ldots, g_n) = \sum_{i=1}^{n} (g_i, 0, \ldots, 0) = \sum_{i=1}^{n} \frac{1}{n} g_i = \frac{1}{n}(g_1 + g_2 + \cdots + g_n). \]

**COROLLARY 2.2.** Let \( G \) be a group and let \( n_1 \geq 2, n_2 \geq 2 \) be integers. Then \( G \) admits an \( n_1 n_2 \)-mean if and only if \( G \) admits an \( n_1 \)-mean and an \( n_2 \)-mean.

### 3. MEANS IN THE CATEGORY \( \mathcal{T}_h \)

Let \( X \) be a connected CW-complex with base point. We prove, with \( n, P \) as in Section 2,

**THEOREM 3.1.** Suppose \( X \) admits an \( n \)-mean \( \mu : X^n \to X \) in \( \mathcal{T}_h \). Then \( X \) is a \( P \)-local commutative \( H \)-space.

**PROOF.** We regard the \( i^{th} \) homotopy group \( \pi_i \) as defining a product-preserving functor from \( \mathcal{T}_h \) to \( G \). Then \( \mu_* = \pi_* \mu : (\pi, X)^n \to \pi, X \) is an \( n \)-mean in \( G \). It follows that \( \pi, X \) is commutative (this is only significant for \( i = 1 \) and \( P \)-local and that \( \mu_* \) has the form (2.1).

Let \( i_1 : X \to X^n \) be the obvious embedding. Then \( (\mu i_1)_* \) is the endomorphism \( g \mapsto \frac{1}{n} g \) of the commutative \( P \)-local group \( \pi, X \). It follows that \( (\mu i_1)_* \) is an automorphism for all \( i \), so that \( \mu i_1 \) is a self-homotopy-equivalence of \( X \). Let \( p : X \to X \) be homotopy inverse to \( \mu i_1 \). Let \( i_2 : X^2 \to X^n \) be the obvious embedding and let \( m = p \mu i_1 : X^2 \to X \). Then it is easy to see that \( m \) is a commutative \( H \)-structure on \( X \). We conclude that \( X \) is a \( P \)-local commutative \( H \)-space.

From Theorem 2.1 we deduce, more easily than in [5],

**THEOREM 3.2.** If a compact, connected polyhedron \( X \) admits an \( n \)-mean for some \( n \geq 2 \), then \( X \) is contractible.

**PROOF.** Since the homotopy groups of \( X \) are \( P \)-local, so are the homology groups \( H_i(X), i \geq 1 \) (see [6]). Now Browder has shown [3] that a compact, connected polyhedron \( X \) which is an \( H \)-space satisfies Poincaré duality. Thus, if \( X \) is not contractible, there exists a positive dimension \( N \) which contains the universal class giving rise to the duality isomorphism \( H_i(X) \cong H_{N-i}(X) \). In particular, \( H_N X = Z \), but this is absurd, since \( Z \) is not divisible by \( n \).

**REMARK 1.** We have not invoked commutativity of the \( H \)-structure in this argument. If we do so, we may apply a theorem of Hubbuck showing that \( X \) would be equivalent to a product of circles, which is also impossible for a non-contractible \( P \)-local space.
**REMARK 2.** Theorem 3.2 is delicate. The n-solenoid is compact and admits an n-mean but is not a polyhedron. The Eilenberg-MacLane space $K(\mathbb{Q}, m)$ is a polyhedron and admits an n-mean for every $n$, but is not compact.

We have not proved—and doubt the truth of—the converse of Theorem 3.1. However, one may readily prove

**THEOREM 3.3.** If $X$ is a $P$-local, connected, commutative, associative $H$-space, then $X$ admits a unique homomorphic n-mean. Further, if the connected $H$-space $(X, m)$ admits a homomorphic n-mean, then $(X, m)$ is commutative and associative.

The case $n = 2$ admits a very neat and precise statement. If $\mu : X^2 \to X$ is a 2-mean on $X$, we define $\rho$ as in the proof of Theorem 3.1 as homotopy inverse to $\mu_1$, and $m = \rho \mu$ is a commutative $H$-structure on the $P$-local space $X$, where $P$ is the family of odd primes. Conversely, if $m : X^2 \to X$ is a commutative $H$-structure on the $P$-local space $X$, we define $\tau$ to be homotopy inverse to $m_1 : X \to X$ (notice that $m_1$ induces doubling on the homotopy groups of $X$ and is therefore a self-homotopy-equivalence). Then $\mu = \tau m$ is a 2-mean on $X$.

**THEOREM 3.4.** The function $\mu \mapsto \rho \mu$ sets up a one-one correspondence between 2-means on the $P$-local connected CW-complex $X$ and commutative $H$-structures on $X$.

**PROOF.** If $m = \rho \mu$, then $\mu_1 = \rho \mu_1 = \rho$, so $\tau$, defined above, is homotopy inverse to $\rho$ and $\tau m = \mu$. If $\tau m = \mu$, then $\tau = \mu_1$ so, again, $\rho$ is homotopy inverse to $\tau$ and $\rho \mu = m$. Thus the function $m \mapsto \tau m$ is inverse to the function $\mu \mapsto \rho \mu$.

**4. THE DUAL STORY**

Whereas the product in a familiar category (like $T_h$, $G$) takes a familiar form essentially independent of the category, the form of the coproduct depends very much on the category in question. The three categories which will come into question here are $T_h$, $G$, and $Ab$, the category of abelian groups.

Let $C$ be a category admitting finite coproducts, we will write $C \vee D$ for the coproduct of $C$ and $D$ in $C$ and $C_n$ for the coproduct of $n$ copies of $C$ in $C$. Obviously, the symmetric group $S_n$ acts on $C_n$, we will write $\nabla : C_n \to C$ for the co-diagonal, which is the morphism that coincides with the identity on each copy of $C$ in $C_n$. Then an n-comean on $C$ is a morphism $\mu : C \to C_n$ such that (i) $\nabla \mu = 1 : C \to C$, and (ii) $\sigma \mu = \mu$, for all $\sigma \in S_n$. We prove

**THEOREM 4.1.** In $G$ only the trivial group admits an n-comean, $n \geq 2$.

**PROOF.** Let $G$ be a non-trivial group and let $g \in G$, $g \neq e$. If $\mu : G \to G_n$ is an n-comean, $n \geq 2$, then it follows from (i) that $\mu g \neq e$. Now $G_n$ is the free product of $n$ copies of $G$, so a non-trivial element of $G_n$ is uniquely expressible as $h_i h_{i-1} \cdots h_{i_1}$, where $G_{(i)}$ is the $i$th copy of $G$ in $G_n$, $h_i \in G_{(i)}$, $h_i \neq e$, and $i_q \neq i_{q+1}$, $q = 1, 2, \ldots, k - 1$. Such an element is obviously moved under any permutation $\sigma$ which moves $i_1$, so that condition (ii) is violated.

**THEOREM 4.2.** In $Ab$, the abelian group $A$ admits an n-comean, $n \geq 2$, if and only if it admits an n-mean. In that case $\mu : A \to A_n$ is given by

$$\mu(a) = \left(\frac{a}{n}, \frac{a}{n}, \ldots, \frac{a}{n}\right).$$

**PROOF.** We note first that, in $Ab$, $C \vee D = C \oplus D$, so that $A_n = A^n$. If $A$ admits an n-mean, then, by Theorem 2.1, it is clear that (4.1) is an n-comean. Suppose conversely that $\mu : A \to A_n$ is an n-comean. It is then plain from (ii) that $\mu(a) = (a, a, \ldots, a)$ for some $a \in A$ such that, by (i), $n a = a$. It remains to show that division by $n$ is unique in $A$. But

$$\mu(na) = (na, na, \ldots, na) = (a, a, \ldots, a),$$

so that $a$ is determined by $na$.
REMARK. Note that the situations for means and comeans are very different. Means in \( G \) coincide with means in \( Ab \), on the other hand, there are no non-trivial comeans in \( G \) but there are non-trivial comeans in \( Ab \), and, moreover, the objects in \( Ab \) admitting \( n \)-comeans coincide with those admitting \( n \)-means.

We now study \( n \)-comeans in \( T_n \). Using the same notation as in Theorem 3.1, we prove

**Theorem 4.3.** Suppose \( X \) is a connected CW-complex admitting an \( n \)-comean \( \mu : X \to X_n \) in \( T_n \), \( n \geq 2 \). Then \( X \) is a simply connected \( P \)-local commutative \( H' \)-space.

**Proof.** Now \( X_n \) is just a bouquet of \( n \) copies of \( X \). Since \( \pi_1 : T_n \to G \) is coproduct-preserving, \( \pi_1 \mu \) is an \( n \)-comean on the fundamental group \( \pi_1 X \), so that, by Theorem 4.1, \( X \) is simply connected. Now the homology groups \( H_i, i \geq 1 \), are coproduct-preserving functors \( T_n \to Ab \), so that, by Theorems 2.1 and 4.2, the homology groups \( H_i X \) are the \( P \)-local. Since \( X \) is simply connected, this implies that \( X \) is \( P \)-local. Finally we adopt a line of reasoning entirely analogous to that in the proof of Theorem 3.1 to conclude that \( X \) admits a commutative \( H' \)-structure \( m : X \to X_2 \). (Notice that, since \( X \) is simply connected, a map \( f : X \to X \) inducing homology isomorphisms is a homotopy equivalence.)

Notice that there are straightforward and valid duals of Theorems 3.3 and 3.4. On the other hand, Theorem 3.2 does not dualize. For example, the Moore space \( M(\mathbb{Z}/2, m) \), \( m \geq 2 \), characterized as the unique simply connected homotopy type with \( H_2 = \mathbb{Z}/2, H_i = 0, i \geq 3 \), is a compact \((m + 1)\)-dimensional polyhedron which admits an \( n \)-comean for every odd \( n \).

**References**
