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OUTER MEASURE ANALYSIS OF TOPOLOGICAL LATTICE PROPERTIES

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ABSTRACT. Let X be a set and $\mathcal{L}$ a lattice of subsets of X such that $\emptyset$, $X \in \mathcal{L}$. $\mathcal{A}(\mathcal{L})$ is the algebra generated by $\mathcal{L}$, $M(\mathcal{L})$ the set of nontrivial, finite, nonnegative, finitely additive measures on $\mathcal{A}(\mathcal{L})$; and $I(\mathcal{L})$ those elements of $M(\mathcal{L})$ which just assume the values zero and one Various subsets of $M(\mathcal{L})$ and $I(\mathcal{L})$ are included which display smoothness and regularity properties.

We consider several outer measures associated with elements of $M(\mathcal{L})$ and relate their behavior to smoothness and regularity conditions as well as to various lattice topological properties In addition, their measurable sets are fully investigated. In the case of two lattices $\mathcal{L}_1$, $\mathcal{L}_2$ with $\mathcal{L}_1 \subset \mathcal{L}_2$, we present consequences of separation properties between the pair of lattices in terms of these outer measures, and further demonstrate the extension of smoothness conditions on $\mathcal{L}_1$ to $\mathcal{L}_2$

KEY WORDS AND PHRASES: Lattice, measure, associated outer measure, regular outer measure, weakly regular, vaguely regular, almost countably compact, complement generated, countably paracompact, delta, normal, semi-separation, separation, coseparation.
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1. INTRODUCTION

Let X be an arbitrary nonempty set and $\mathcal{L}$ a lattice of subsets of X with $\emptyset$, $X \in \mathcal{L}$. $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$ and $M(\mathcal{L})$ the set of nontrivial, finite, nonnegative, finitely additive measures on $\mathcal{A}(\mathcal{L})$. We consider various specialized subsets of $M(\mathcal{L})$ and introduce several outer measures associated with them. Extending the work done in [3,4], we further investigate the interplay of these outer measures with the various subsets of $M(\mathcal{L})$ as well as with lattice topological properties. Frequently, this is carried out under the assumption of regularity on one of the outer measures.

In addition, we analyze in detail the situation when $\mathcal{L}_1 \subset \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices of subsets of X When separation conditions are satisfied between these two lattices, the behavior of the associated outer measures reflects very strongly on the lattices. Our results here extend those obtained for zero-one valued measures in [5,6,8].

We begin in section 2 with a brief review of some relevant facts and notations that will be used throughout the paper. In addition, a few new basic results on the associated outer measures are established. In section 3, we investigate the effects of lattice and lattice topological properties on the outer measures, and, in turn, the latter's behavior in characterizing certain subsets of $M(\mathcal{L})$. Section 4 is mainly concerned with the case of $\mathcal{L}_1 \subset \mathcal{L}_2$and the effects of separation properties between them on the outer measures.

Further related matters can be found in [2,3,4].

2. BACKGROUND AND NOTATIONS

In this section we introduce the notation and terminology that will be used throughout the paper. They are mostly standard and we review the more important ones for the readers' convenience. For further details see [3,4,6].

Let X be an arbitrary nonempty set and $\mathcal{L}$ a lattice of subsets of X such that $\emptyset, X \in \mathcal{L}$ $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$, and $M(\mathcal{L})$ the set of nontrivial, nonnegative, finitely additive, finite measures on $\mathcal{A}(\mathcal{L})$. We denote by $M_\sigma(\mathcal{L})$ those measures in $M(\mathcal{L})$ that are σ-smooth on $\mathcal{L}$, namely, if $L_n \downarrow \emptyset$, $L_n \in \mathcal{L}$ then $\mu(L_n) \to 0$ $M^\circ(\mathcal{L})$ designates those $\mu \in M(\mathcal{L})$ that are strongly σ-smooth on $\mathcal{L}$, i e., $L_n \downarrow L$, $L, L_n \in \mathcal{L}$ implies $\mu(L_n) \downarrow \mu(L)$. Also, $M^\sigma(\mathcal{L})$ denotes those $\mu \in M(\mathcal{L})$ that are σ-smooth on $\mathcal{A}(\mathcal{L})$, so $A_n \downarrow \emptyset$, $A_n \in \mathcal{A}(\mathcal{L})$ implies $\mu(A_n) \to 0$, this is equivalent to μ being countably additive.

In addition, $M_R(\mathcal{L})$ denotes those $\mu \in M(\mathcal{L})$ that are $\mathcal{L}$-regular: if for any $A \in \mathcal{A}(\mathcal{L})$, $\mu(A) = \sup\{\mu(L) | L \subset A, L \in \mathcal{L}\}$ Let $M_R^\sigma(\mathcal{L}) = M_R(\mathcal{L}) \cap M_\sigma(\mathcal{L})$, clearly $\mu \in M_R^\sigma(\mathcal{L})$ implies $\mu \in M^\sigma(\mathcal{L})$. If the measures just assume the values zero and one we denote the above sets by I's replacing their corresponding M's.

For any set $E \subset X$, $E' = X - E$, and $\mathcal{L}' = \{L' | L \in \mathcal{L}\}$ is the complementary lattice to $\mathcal{L}$ We denote by $\delta(\mathcal{L})$ the lattice of all countable intersections of sets from $\mathcal{L}$, and $\mathcal{L}$ is *delta* if $\delta(\mathcal{L}) = \mathcal{L}$, i.e., $\mathcal{L}$ is closed under countable intersections. We shall utilize the following lattice topological notions as well as their measure characterizations (see [6]): $\mathcal{L}$ is *normal* if whenever A, $B \in \mathcal{L}$ such that $A \cap B = \emptyset$, there exist C, $D \in \mathcal{L}$ such that $A \subset C'$, $B \subset D'$ and $C' \cap D' = \emptyset$; $\mathcal{L}$ is *countably paracompact* if for every sequence $\{A_n\}$ of sets of $\mathcal{L}$ with $A_n \downarrow \emptyset$, there exists a sequence $\{B_n\}$ in $\mathcal{L}$ such that for all n, $A_n \subset B_n'$ and $B_n' \downarrow \emptyset$; and $\mathcal{L}$ is *complement generated* if for every $L \in \mathcal{L}$ there exist $A_n \in \mathcal{L}$ such that $L = \bigcap_{n=1}^{\infty} A_n'$. Furthermore, we define for $\mu \in M(\mathcal{L})$ and $E \subset X$,

$$\begin{aligned}
\mu'(E) &= \inf\{\mu(L') | E \subset L', L \in \mathcal{L}\} \\
\mu''(E) &= \inf\left\{\sum_{i=1}^{\infty} \mu(L_i') | E \subset \bigcup_{i=1}^{\infty} L_i', L_i \in \mathcal{L}\right\} \\
\tilde{\mu}(E) &= \inf\{\mu(L) | E \subset L, L \in \mathcal{L}\}
\end{aligned}$$

Clearly all these set functions are nonnegative, monotonic, and vanish on the null set; μ' and $\tilde{\mu}$ are finitely subadditive while μ'' is countably subadditive. We shall simply refer to them as associated outer measures of μ. It is also clear that $\mu'(X) = \tilde{\mu}(X) = \mu(X)$. If $\mu \in M_\sigma(\mathcal{L})$ then $\mu''(X) = \mu(X)$. In fact, if $\mu \in I(\mathcal{L})$ and if $\mu \notin I_\sigma(\mathcal{L})$ then $\mu'' = 0$.

If μ and ν are measures or outer measures we write $\mu \leq \nu(\mathcal{L})$ whenever $\mu(L) \leq \nu(L)$ for all $L \in \mathcal{L}$. Using this notation, it is easy to see that for $\mu \in M_\sigma(\mathcal{L})$, $\mu \leq \mu''(\mathcal{L})$. Trivially, $\mu'' \leq \mu'$, and $\mu = \mu'(\mathcal{L})$ if and only if $\mu \in M_R(\mathcal{L})$.

Consider any two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ of subsets of X. We say $\mathcal{L}_1$ *semi-separates* $\mathcal{L}_2$ if for $A_1 \in \mathcal{L}_1$, $A_2 \in \mathcal{L}_2$ with $A_1 \cap A_2 = \emptyset$, there exists $B_1 \in \mathcal{L}_1$ such that $A_2 \subset B_1 \subset A_1'$; $\mathcal{L}_1$ *separates* $\mathcal{L}_2$ if for A_2, $B_2 \in \mathcal{L}_2$ with $A_2 \subset B_2'$, there exists A_1, $B_1 \in \mathcal{L}_1$ such that $A_2 \subset A_1 \subset B_1' \subset B_2'$; and $\mathcal{L}_1$ *coseparates* $\mathcal{L}_2$ if for A_2, $B_2 \in \mathcal{L}_2$ with $A_2 \subset B_2'$, there exist A_1, $B_1 \in \mathcal{L}_1$ such that $A_2 \subset A_1' \subset B_1 \subset B_2'$.

Finally, if ν is an outer measure defined on all subsets of X (finitely or countably subadditive) then S_ν denotes the ν-measurable sets.

We collect a number of important lattice measure results that will be utilized throughout the paper

1 If $\mu \in M(\mathcal{L})$ then there exists a $\nu \in M_R(\mathcal{L})$ such that $\mu \leq \nu(\mathcal{L})$ and $\mu(X) = \nu(X)$ [7]

2. Let $\mathcal{L}$ be normal and $\mu \in M(\mathcal{L})$. Suppose $\mu \leq \nu(\mathcal{L})$, $\nu \in M_R(\mathcal{L})$ and $\mu(X) = \nu(X)$. Then for $L \in \mathcal{L}$, $\nu(L') = \sup\{\mu(\tilde{L}) | \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$. [4]

3 Let $\mathcal{L}$ be normal and $\mu \in M_\sigma(\mathcal{L})$. Suppose $\mu \leq \nu(\mathcal{L})$, $\nu \in M_R(\mathcal{L})$ and $\mu(X) = \nu(X)$ Then $\nu \in M_\sigma(\mathcal{L}')$. [4]

4 Let $\mathcal{L}$ be normal and $\mu \in M(\mathcal{L})$, $\nu \in M_R(\mathcal{L})$. If $\mu \leq \nu(\mathcal{L})$ and if $\mu(X) = \nu(X)$ then $\mu \leq \nu = \nu' = \mu'$ on $\mathcal{L}$ [4]

5 Let $\mathcal{L}$ be normal and countably paracompact. If $\mu, \nu \in M(\mathcal{L})$ such that $\mu \leq \nu(\mathcal{L})$ and $\mu(X) = \nu(X)$ then $\mu \in M_\sigma(\mathcal{L})$ implies $\nu \in M_\sigma(\mathcal{L})$. [8]

Next, we consider any two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$ Then we have

6. Any $\mu \in M_R(\mathcal{L}_1)$ can be extended to a $\nu \in M_R(\mathcal{L}_2)$. [1]

7 If $\nu \in M_R(\mathcal{L}_2)$ then ν restricted to $\mathcal{A}(\mathcal{L}_1)$ (we denote this restriction by $\nu|_{\mathcal{L}_1}$ or even just $\nu|$ if the lattices involved are clear) belongs to $M_R(\mathcal{L}_1)$ if $\mathcal{L}_1$ semi-separates $\mathcal{L}_2$ [2,6]

THEOREM 2.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$, and let $\mathcal{L}_2$ be $\mathcal{L}_1$-countably bounded Suppose $\rho \in M(\mathcal{L}_2)$ extends $\mu \in M_\sigma(\mathcal{L}_1)$, and $\mu \leq \nu(\mathcal{L}_1)$, $\nu \in M_R(\mathcal{L}_1)$ where $\mu(X) = \nu(X)$ In addition, suppose $\rho \leq \lambda(\mathcal{L}_2)$, $\lambda \in M_R(\mathcal{L}_2)$ where $\rho(X) = \lambda(X)$ Then the following are true:

(a) $\rho \in M_\sigma(\mathcal{L}_2)$

(b) If $\mathcal{L}_1$ is normal and countably paracompact, then $\nu \in M_R^\sigma(\mathcal{L}_1)$.

(c) If $\mathcal{L}_2$ is normal and countably paracompact, then $\lambda \in M_R^\sigma(\mathcal{L}_2)$.

(d) If $\mathcal{L}_1$ is normal and complement generated and if $\mathcal{L}_2$ is normal and countably paracompact, then $\lambda|_{\mathcal{L}_1} = \nu$.

PROOF.

(a) Let $B_n \in \mathcal{L}_2$ such that $B_n \downarrow \emptyset$ Since $\mathcal{L}_2$ is $\mathcal{L}_1$-countably bounded, there exist $A_n \in \mathcal{L}_1$ such that $B_n \subset A_n \downarrow \emptyset$ Consequently,

$$\rho(B_n) \leq \rho(A_n) = \mu(A_n) \downarrow 0.$$

(b) By statement (5), $\nu \in M_R^\sigma(\mathcal{L}_1)$.

(c) Again follows by statement (5).

(d) Since $\lambda \in M_R^\sigma(\mathcal{L}_2)$, $\lambda|_{\mathcal{L}_1} \in M_\sigma(\mathcal{L}_1')$. Hence, $\lambda|_{\mathcal{L}_1} \in M_R^\sigma(\mathcal{L}_1)$ since $\mathcal{L}_1$ is complement generated. Then by normality, $\lambda|_{\mathcal{L}_1} = \nu$ (see remark). ✥

REMARK. In the proof of (d) we have assumed the following two facts·

- If $\mathcal{L}$ is complement generated then $\mathcal{L}$ is countably paracompact.
- If $\mathcal{L}$ is normal and if $\mu \in M(\mathcal{L})$, $\nu_1, \nu_2 \in M_R(\mathcal{L})$ with $\mu \leq \nu_1(\mathcal{L})$, $\mu \leq \nu_2(\mathcal{L})$ and $\mu(X) = \nu_1(X) = \nu_2(X)$, then $\nu_1 = \nu_2$.

The first fact is elementary, the second can be found in [4].

DEFINITION 2.1. A measure $\mu \in M(\mathcal{L})$ is *weakly regular* if, for any $L \in \mathcal{L}$,

$$\mu(L') = \sup\{\mu'(\tilde{L}) | \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}.$$

We denote the set of weakly regular elements of $M(\mathcal{L})$ by $M_W(\mathcal{L})$. Clearly $M_R(\mathcal{L}) \subset M_W(\mathcal{L})$, and if $\mathcal{L}$ is normal then $M_R(\mathcal{L}) = M_W(\mathcal{L})$ (see [4]).

We recall that the lattice $\mathcal{L}$ is *almost countably compact* if for any $\mu \in I_R(\mathcal{L}')$, $\mu \in I_\sigma(\mathcal{L})$ It follows readily that if $\mathcal{L}$ is almost countably compact then $\mu \in M_R(\mathcal{L}')$ implies $\mu \in M_\sigma(\mathcal{L})$.

3. PROPERTIES OF ASSOCIATED OUTER MEASURES

This section begins with an enumeration of several known properties of the associated outer measures introduced in Section 2 We shall then develop new properties and characterizations

THEOREM 3.1.

Let $\mu \in M(\mathcal{L})$. Then $E \in S_{\mu'}$ if and only if one of the following is true.

(a) $\mu'(X) = \mu'(E) + \mu'(E')$

(b) $\mu_i(E) = \mu'(E)$, where $\mu_i(E) = \sup\{\mu(L) | L \subset E, L \in \mathcal{L}\}$

2. If $\mu \in M_\sigma(\mathcal{L})$ then $\mu'' \leq \mu'$, $\mu''(X) = \mu(X)$, and $\mu \leq \mu''(\mathcal{L})$
3 If $\mu \in M_R^\sigma(\mathcal{L})$ then $\mu = \mu'' = \mu'$ on $\mathcal{L}$
4. If $\mu \in M(\mathcal{L})$ then $S_{\mu'} \cap \mathcal{L} = \{L \in \mathcal{L} | \mu(L) = \mu'(L)\}$, and consequently $\mathcal{L} \subset S_{\mu'}$ if and only if $\mu \in M_R(\mathcal{L})$.
5. If $\mathcal{L}$ is a delta lattice and if $\mu \in M^\circ(\mathcal{L})$, then μ is countably subadditive on $\mathcal{L}'$, $\mu' = \mu''$, whence $S_{\mu'} = S_{\mu''}$

PROOF. See proof of Theorem 3.2(a) and [4].

DEFINITION 3.1. Let ν be a finite outer measure (finite- or countably subadditive). Then ν is *regular* if for every $E \subset X$, there exists $M \in S_\nu$ such that $E \subset M$ and $\nu(E) = \nu(M)$ Clearly ν is regular if it assumes only the values 0 and 1. In addition, if ν is regular then $E \in S_\nu$ if and only if $\nu(X) = \nu(E) + \nu(E')$; also if ν is a regular countably subadditive outer measure and if $E_n \uparrow$, $E_n \subset X$, then

$$\nu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \nu(E_n).$$

THEOREM 3.2. (a) If $\mu \in M^\circ(\mathcal{L})$ then $\mu' = \mu''(\mathcal{L}')$. (b) Let $\mu \in M_\sigma(\mathcal{L})$. If $\mu' = \mu''(\mathcal{L}')$ and if μ'' is a regular outer measure, then $\mu \in M^\circ(\mathcal{L})$

PROOF. (a) This is proved in [4] under assumptions. We give a direct proof here without any further assumptions Firstly, we shall show that if $\mu \in M^\circ(\mathcal{L})$ and if $\bigcup_{i=1}^{\infty} L_i' \in \mathcal{L}'$ for every sequence $\{L_i'\}$, $L_i \in \mathcal{L}$ for all i, then

$$\mu\left(\bigcup_{i=1}^{\infty} L_i'\right) \leq \sum_{i=1}^{\infty} \mu(L_i'),$$

i.e., μ is countably subadditive on $\mathcal{L}'$.
Suppose there exists a sequence $\{L_i'\}$, $L_i \in \mathcal{L}$, such that

$$\mu\left(\bigcup_{i=1}^{\infty} L_i'\right) > \sum_{i=1}^{\infty} \mu(L_i') = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(L_i') \geq \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} L_i'\right).$$

We have

$$\bigcup_{i=1}^{n} L_i' \uparrow \bigcup_{i=1}^{\infty} L_i',$$

therefore, $\mu \in M^\circ(\mathcal{L})$ implies

$$\mu\left(\bigcup_{i=1}^{\infty} L_i'\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} L_i'\right),$$

a contradiction.

Now let $L \in \mathcal{L}$ and let $L_n \in \mathcal{L}$ such that $L' \subset \bigcup_{n=1}^{\infty} L_n'$ Then $L' = \bigcup_{n=1}^{\infty} (L_n' \cap L')$ Using the preceding result, it follows that

$$\mu(L') \leq \sum_{n=1}^{\infty} \mu(L_n' \cap L') \leq \sum_{n=1}^{\infty} \mu(L_n').$$

Therefore,

$$\mu(L') \leq \inf\left\{\sum_{n=1}^{\infty} \mu(L_n') | L' \subset \bigcup_{n=1}^{\infty} L_n', L_n \in \mathcal{L}\right\} = \mu''(L').$$

Hence $\mu \leq \mu''(\mathcal{L}')$ Since $\mu'' \leq \mu(\mathcal{L}')$, we get $\mu = \mu''(\mathcal{L}')$ and thus $\mu' = \mu''(\mathcal{L}')$

(b) see [4] ✥

We now consider several new results

THEOREM 3.3. (a) Let $\mu \in M(\mathcal{L})$. If $\mu \in M_R(\mathcal{L})$ then $\mu' \leq \tilde{\mu}$. Conversely, if $\mu' \leq \tilde{\mu}(\mathcal{L})$ then $\mu \in M_R(\mathcal{L})$

(b) $\mu \in M^\circ(\mathcal{L})$ if and only if $\tilde{\mu}\left(\bigcap_{n=1}^{\infty} L_n\right) = \inf_n \mu(L_n)$ where $L_n \downarrow$, $L_n \in \mathcal{L}$

(c) Let $\mu \in M_\sigma(\mathcal{L})$ Suppose for $E \subset X$,

$$\mu''(E) = \sup\{\mu''(D) | D \subset E, D \in \delta(\mathcal{L})\}$$

and suppose μ' is countably subadditive on $\mathcal{L}'$. Then $\mu' = \mu''$

PROOF. (a) By definition, given $\epsilon > 0$ there exists $L \in \mathcal{L}$, $E \subset L$ such that

$$\tilde{\mu}(E) > \mu(L) - \epsilon.$$

Since $\mu \in M_R(\mathcal{L})$, there exists $\tilde{L} \in \mathcal{L}$, $L \subset \tilde{L}'$ such that

$$\mu(L) > \mu\left(\tilde{L}'\right) - \epsilon.$$

Consequently,

$$\tilde{\mu}(E) > \mu\left(\tilde{L}'\right) - 2\epsilon = \mu'\left(\tilde{L}'\right) - 2\epsilon \geq \mu'(E) - 2\epsilon$$

since $E \subset L \subset \tilde{L}'$. Hence $\tilde{\mu}(E) \geq \mu'(E)$ and thus $\mu' \leq \tilde{\mu}$.

For $\mu \in M(\mathcal{L})$, $\mu \leq \mu'(\mathcal{L})$; hence by hypothesis, $\mu \leq \mu' \leq \tilde{\mu}$ on $\mathcal{L}$. But clearly $\mu = \tilde{\mu}(\mathcal{L})$ by definition Therefore $\mu = \mu'(\mathcal{L})$ or equivalently, $\mu \in M_R(\mathcal{L})$.

(b) If the condition is satisfied and if $\bigcap_{n=1}^{\infty} L_n = L \in \mathcal{L}$, then

$$\mu(L) = \tilde{\mu}(L) = \inf_n \mu(L_n) = \lim_{n\to\infty} \mu(L_n).$$

Thus $\mu \in M^\circ(\mathcal{L})$.

Conversely, suppose $\mu \in M^\circ(\mathcal{L})$. Then by definition of $\tilde{\mu}$,

$$\tilde{\mu}\left(\bigcap_{n=1}^{\infty} L_n\right) = \inf\left\{\mu(L) \middle| \bigcap_{n=1}^{\infty} L_n \subset L, L \in \mathcal{L}\right\}.$$

Since $\bigcap_{n=1}^{\infty} L_n \subset L_n$ for any n,

$$\tilde{\mu}\left(\bigcap_{n=1}^{\infty} L_n\right) \leq \mu(L_n) \quad \text{for any} \quad n.$$

Therefore,

$$\tilde{\mu}\left(\bigcap_{n=1}^{\infty} L_n\right) \leq \inf_n \mu(L_n) = \lim_{n\to\infty} \mu(L_n).$$

Suppose

$$\lim_{n\to\infty} \mu(L_n) > \tilde{\mu}\left(\bigcap_{n=1}^{\infty} L_n\right) \quad \text{for} \quad L_n \downarrow, L_n \in \mathcal{L}.$$

Then there exists $\epsilon > 0$ such that

$$\lim_{n\to\infty} \mu(L_n) > \tilde{\mu}\left(\bigcap_{n=1}^{\infty} L_n\right) + \epsilon.$$

Also there exists $L \in \mathcal{L}$, $\bigcap_{n=1}^{\infty} L_n \subset L$ such that

$$\mu(L) \leq \tilde{\mu}\left(\bigcap_{n=1}^{\infty} L_n\right) + \epsilon/2.$$

Now, $\bigcap_{n=1}^{\infty} (L_n \cup L) = L$ and $L_n \cup L \downarrow$. Hence $\mu(L_n \cup L) \to \mu(L)$ since $\mu \in M^{\circ}(\mathcal{L})$ It then follows that

$$\tilde{\mu}\left(\bigcap_{n=1}^{\infty} L_n\right) + \epsilon/2 > \mu(L) = \lim_{n\to\infty} \mu(L_n \cup L) \geq \lim_{n\to\infty} \mu(L_n) > \tilde{\mu}\left(\bigcap_{n=1}^{\infty} L_n\right) + \epsilon,$$

a contradiction. Thus

$$\lim_{n\to\infty} \mu(L_n) = \tilde{\mu}\left(\bigcap_{n=1}^{\infty} L_n\right).$$

(c) By hypothesis, given $\epsilon > 0$ there exists $D \in \delta(\mathcal{L})$, $D \subset E$ such that

$$\mu''(E) - \epsilon < \mu''(D).$$

Also there exist $L_i \in \mathcal{L}$, $E \subset \bigcup_{i=1}^{\infty} L_i'$ such that

$$\sum_{i=1}^{\infty} \mu(L_i') - \epsilon < \mu''(E).$$

Consequently,

$$\begin{aligned}
\mu''(E) \geq \mu''(D) > \mu''(E) - \epsilon > \sum_{i=1}^{\infty} \mu(L_i') - 2\epsilon = \sum_{i=1}^{\infty} \mu'(L_i') - 2\epsilon \\
\geq \mu'\left(\bigcup_{i=1}^{\infty} L_i'\right) - 2\epsilon \quad \text{since } \mu' \text{ is countably subadditive on } \mathcal{L}' \\
\geq \mu'(E) - 2\epsilon \qquad \text{since } E \subset \bigcup_{i=1}^{\infty} L_i'
\end{aligned}$$

Hence $\mu''(E) \geq \mu'(E)$ and thus, $\mu'' \geq \mu'$. But $\mu'' \leq \mu'$, hence $\mu'' = \mu'$ ✥

If $\mu \in M^{\sigma}(\mathcal{L})$ then μ^* denotes the usual induced outer measure, i.e , for $E \subset X$

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) | E \subset \bigcup_{i=1}^{\infty} A_i, A_i \in \mathcal{A}(\mathcal{L}) \text{ for all } i\right\}$$

Clearly, if $\mu \in M^{\sigma}(\mathcal{L})$ then $\mu^* \leq \mu'' \leq \mu'$; moreover if $\mathcal{L}$ is a delta lattice and if $\mu \in M_R^{\sigma}(\mathcal{L})$, then $\mu^* = \mu' = \mu''$.

We wish to consider certain consequences of equalities between a measure and one of its associated outer measures. Firstly, we define

DEFINITION 3.2. A measure $\mu \in M_{\sigma}(\mathcal{L})$ is *vaguely regular* if for any $L \in \mathcal{L}$,

$$\mu(L') = \sup\{\mu''(\tilde{L}) | \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}.$$

We denote by $M_V(\mathcal{L})$ the set of vaguely regular measures

THEOREM 3.4. (a) Let $\mu \in M_W(\mathcal{L})$ and $\nu \in M_R(\mathcal{L})$ Suppose $\mu \leq \nu(\mathcal{L})$ and $\mu(X) = \nu(X)$ If $\mu' = \nu(\mathcal{L})$ then $\mu = \nu$.

(b) Let $\mu \in M_V(\mathcal{L})$ and $\nu \in M_R(\mathcal{L})$ Suppose $\mu \leq \nu(\mathcal{L})$ and $\mu(X) = \nu(X)$. If $\mu'' = \nu(\mathcal{L})$ then $\mu = \nu$ and $\nu \in M_R^\sigma(\mathcal{L})$

(c) Let $\mu \in M_\sigma(\mathcal{L})$. If $\mu = \mu''(\mathcal{L})$ and if μ'' is a regular outer measure, then

$$\mathcal{A}(\mathcal{L}) \subset S_{\mu''}, \quad \mu = \mu'' \quad \text{on} \quad \mathcal{A}(\mathcal{L}), \quad \text{and} \quad \mu \in M^\sigma(\mathcal{L}).$$

(d) If $\mu \in M_\sigma(\mathcal{L})$ and if μ'' is a regular outer measure, then $S_{\mu'} \subset S_{\mu''}$ and $\mu' = \mu''$ on $S_{\mu'}$

PROOF. (a) By weak regularity, for any $L \in \mathcal{L}$,

$$\begin{aligned}\mu(L') &= \sup\{\mu'(\tilde{L}) | \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\} \\ &= \sup\{\nu(\tilde{L}) | \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\} \quad \text{by hypothesis} \\ &= \nu(L') \quad \text{since} \quad \nu \in M_R(\mathcal{L}).\end{aligned}$$

Therefore $\mu = \nu$

(b) By vague regularity, for any $L \in \mathcal{L}$,

$$\begin{aligned}\mu(L') &= \sup\{\mu''(\tilde{L}) | \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\} \\ &= \sup\{\nu(\tilde{L}) | \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\} \quad \text{by hypothesis} \\ &= \nu(L') \quad \text{since} \quad \nu \in M_R(\mathcal{L}).\end{aligned}$$

Therefore $\nu = \mu \in M_V(\mathcal{L}) \subset M_\sigma(\mathcal{L})$, and thus $\nu \in M_R^\sigma(\mathcal{L})$.

(c) Let $L \in \mathcal{L}$, then

$$\mu''(X) = \mu(X) = \mu(L) + \mu(L') \geq \mu''(L) + \mu''(L') \geq \mu''(X)$$

since $\mu = \mu''(\mathcal{L})$ and $\mu'' \leq \mu(\mathcal{L}')$. Hence

$$\mu''(X) = \mu''(L) + \mu''(L').$$

Consequently, $L \in S_{\mu''}$ for any $L \in \mathcal{L}$ since μ'' is regular. Thus $\mathcal{L} \subset S_{\mu''}$ and hence $\mathcal{A}(\mathcal{L}) \subset S_{\mu''}$

Now μ'' is countably additive on $S_{\mu''}$ and equals μ on $\mathcal{L}$; therefore, $\mu = \mu''$ on $\mathcal{A}(\mathcal{L})$ Finally, since μ'' is countably additive on $S_{\mu''} \supset \mathcal{A}(\mathcal{L})$, $\mu \in M^\sigma(\mathcal{L})$.

(d) Let $E \in S_{\mu'}$. Since μ'' is regular, it suffices to show that

$$\mu''(X) = \mu''(E) + \mu''(E').$$

Now

$$\begin{aligned}\mu(X) &= \mu''(X) \quad \text{since } \mu \in M_\sigma(\mathcal{L}) \\ &\leq \mu''(E) + \mu''(E') \quad \text{since } \mu'' \text{ is finitely subadditive} \\ &\leq \mu'(E) + \mu'(E') \quad \text{since } \mu'' \leq \mu' \\ &= \mu'(X) \quad \text{since } E \in S_{\mu'} \\ &= \mu(X).\end{aligned}$$

Therefore, $\mu''(X) = \mu''(E) + \mu''(E')$. Hence $E \in S_{\mu''}$, and thus $S_{\mu'} \subset S_{\mu''}$

Also by Theorem 3 1(1.b),

$$\begin{aligned}\mu'(E) = \mu_i(E) &= \sup\{\mu(L) | L \subset E, L \in \mathcal{L}\} \\ &\leq \sup\{\mu''(L) | L \subset E, L \in \mathcal{L}\} \quad \text{since } \mu \leq \mu''(\mathcal{L}) \text{ for } \mu \in M_\sigma(\mathcal{L}) \\ &\leq \mu''(E).\end{aligned}$$

But $\mu'' \leq \mu'$, hence $\mu' = \mu''$ on $S_{\mu'}$. ✥

THEOREM 3.5. (a) Let $\mu \in M_\sigma(\mathcal{L})$. If $\mathcal{L}$ is delta-normal then $\mu' = \mu''(\mathcal{L})$

(b) Let $\mu \in M_\sigma(\mathcal{L})$. If $\mathcal{L}$ is a delta-normal lattice and if $\mu \leq \nu(\mathcal{L})$ where $\nu \in M_R(\mathcal{L})$ and $\mu(X) = \nu(X)$, and if μ'' is a regular outer measure, then $\nu \in M^\circ(\mathcal{L}')$

PROOF. (a) see [4].

(b) Recall earlier that $\mu \leq \nu(\mathcal{L})$ and $\mu(X) = \nu(X)$ implies $\mu \leq \nu = \nu' = \mu' = \mu''(\mathcal{L})$ since $\mathcal{L}$ is delta-normal Suppose $\nu \notin M^\circ(\mathcal{L}')$, i.e.,

$$\nu(L') \neq \inf_n \nu(L'_n)$$

for some sequence $\{L'_n\} \in \mathcal{L}'$ and some $L' \in \mathcal{L}'$, where $L'_n \downarrow L'$. Accordingly, there exists $\epsilon > 0$ such that

$$\nu(L'_n) - \epsilon > \nu(L') \text{ for all } n.$$

Thus $\nu(L_n) + \epsilon < \nu(L)$. Now μ'' is a regular outer measure and $L_n \uparrow L$, therefore

$$\lim_{n\to\infty} \mu''(L_n) = \mu''(L).$$

Hence by hypothesis,

$$\nu(L) - \epsilon > \nu(L_n) = \mu''(L_n) \text{ for all } n.$$

Consequently,

$$\nu(L) - \epsilon \geq \lim_{n\to\infty} \mu''(L_n) = \mu''(L) = \nu(L),$$

a contradiction. So $\nu \in M^\circ(\mathcal{L}')$. ✥

We next consider two lattices of subsets of X, $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1 \subset \mathcal{L}_2$, and give a necessary and sufficient condition for a regular measure extension to be $\mathcal{L}_1$-regular on $\mathcal{L}'_2$.

THEOREM 3.6. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ be lattices of subsets of X. Let $\nu \in M_R(\mathcal{L}_2)$ be an extension of $\mu \in M_R(\mathcal{L}_1)$. Then ν is $\mathcal{L}_1$-regular on $\mathcal{L}'_2$ if and only if $\nu = \mu'(\mathcal{L}_2)$

PROOF. Suppose ν is $\mathcal{L}_1$-regular on $\mathcal{L}'_2$, i.e., $\nu(L'_2) = \sup\{\nu(L_1) | L_1 \subset L'_2, L_1 \in \mathcal{L}_1\}$ for all $L_2 \in \mathcal{L}_2$ Then given $\epsilon > 0$ there exists $L_1 \in \mathcal{L}_1$, $L_1 \subset L'_2$ such that

$$\nu(L'_2) - \epsilon < \nu(L_1) = \mu(L_1).$$

Accordingly,

$$\nu(L_2) + \epsilon > \mu(L'_1) = \mu'(L'_1) > \mu'(L_2).$$

Hence $\nu \geq \mu'(\mathcal{L}_2)$ But

$$\mu'(L_2) = \inf \mu(L'_1) = \inf \nu(L'_1) \geq \nu(L_2), \quad \text{where} \quad L_2 \subset L'_1, \quad L_1 \in \mathcal{L}_1.$$

Thus $\mu' \geq \nu(\mathcal{L}_2)$

Conversely, suppose $\nu = \mu'(\mathcal{L}_2)$. Let $L_2 \in \mathcal{L}_2$ Then

$$\nu(L_2) = \inf\{\mu(L'_1) | L_2 \subset L'_1, L_1 \in \mathcal{L}_1\}.$$

Consequently,

$$\begin{aligned} \nu(L'_2) &= \mu(X) - \inf\{\mu(L'_1) | L_2 \subset L'_1, L_1 \in \mathcal{L}_1\} \\ &= \sup\{\mu(L_1) | L_1 \subset L'_2, L_1 \in \mathcal{L}_1\} \\ &= \sup\{\nu(L_1) | L_1 \subset L'_2, L_1 \in \mathcal{L}_1\}. \end{aligned}$$

In other words, ν is $\mathcal{L}_1$-regular on $\mathcal{L}'_2$ ✥

THEOREM 3.7. Suppose $\delta(\mathcal{L}')$ separates $\mathcal{L}$ If $\mu \in M_W(\mathcal{L}) \cap M_\sigma(\mathcal{L}')$ then $\mu \in M_R(\mathcal{L})$

PROOF. Let $L \in \mathcal{L}$ and $\epsilon > 0$. Then, since $\mu \in M_W(\mathcal{L})$ there exists $\tilde{L} \in \mathcal{L}$, $\tilde{L} \subset L'$ such that $\mu(L') - \epsilon < \mu'(\tilde{L})$. Now since $\delta(\mathcal{L}')$ separates $\mathcal{L}$, there exist $A_i, B_j \in \mathcal{L}$ such that

$$\tilde{L} \subset \bigcap_{i=1}^{\infty} A_i' \in \delta(\mathcal{L}'), \quad L \subset \bigcap_{j=1}^{\infty} B_j' \in \delta(\mathcal{L}'), \quad \text{and} \quad \bigcap_{i,j} A_i' \cap B_j' = \emptyset.$$

We may assume $A_i' \cap B_j' \downarrow$. Hence, since $\mu \in M_\sigma(\mathcal{L}')$ there exists N such that for $i, j \geq N$, $\mu(A_i' \cap B_j') < \epsilon$ Now

$$\begin{aligned}
\mu(A_i' \cup B_j') &= \mu(A_i') + \mu(B_j') - \mu(A_i' \cap B_j') \\
&\geq \mu(A_i') + \mu(B_j') - \epsilon \quad \text{for } i, j \geq N \\
&\geq \mu'(\tilde{L}) + \mu'(L) - \epsilon \\
&\geq \mu(L') + \mu'(L) - 2\epsilon \\
&= \mu'(L') + \mu'(L) - 2\epsilon.
\end{aligned}$$

Therefore,

$$\mu'(X) = \mu(X) \geq \mu(A_i' \cup B_j') \geq \mu'(L') + \mu'(L) - 2\epsilon.$$

Hence by the arbitrariness of $\epsilon > 0$ and by Theorem 3.1(1.a), $L \in S_{\mu'}$ for any $L \in \mathcal{L}$ Thus $\mathcal{L} \subset S_{\mu'}$ and by Theorem 3.1(4), $\mu \in M_R(\mathcal{L})$. ✣

Finally, we improve on Theorem 3.5(a).

THEOREM 3.8. Let $\mathcal{L}_1 \subset \mathcal{L}_2$. Suppose $\mathcal{L}_1$ is a delta lattice and $\mathcal{L}_1$ semi-separates $\mathcal{L}_2$ If $\mu \in M_\sigma(\mathcal{L}_1)$ and if $\mu' = \mu''(\mathcal{L}_1)$ then $\mu' = \mu''(\mathcal{L}_2)$.

PROOF. Let $L_2 \in \mathcal{L}_2$. Since $\mu \in M_\sigma(\mathcal{L}_1)$, given $\epsilon > 0$ there exist $L_i \in \mathcal{L}_1$ such that

$$L_2 \subset \bigcup_{i=1}^{\infty} L_i' \quad \text{and} \quad \sum_{i=1}^{\infty} \mu(L_i') - \epsilon < \mu''(L_2).$$

Hence $L_2 \cap \left(\bigcap_{i=1}^{\infty} L_i\right) = \emptyset$ where $\bigcap_{i=1}^{\infty} L_i \in \mathcal{L}_1$ since $\mathcal{L}_1$ is delta. By semi-separation there exists $A_1 \in \mathcal{L}_1$ such that

$$L_2 \subset A_1 \subset \bigcup_{i=1}^{\infty} L_i'.$$

Therefore

$$\mu'(L_2) \leq \mu'(A_1) = \mu''(A_1) \leq \mu''\left(\bigcup_{i=1}^{\infty} L_i'\right) \leq \sum_{i=1}^{\infty} \mu(L_i') < \mu''(L_2) + \epsilon;$$

and so $\mu'(L_2) \leq \mu''(L_2)$ for any $L_2 \in \mathcal{L}_2$. But $\mu'' \leq \mu'$, thus $\mu' = \mu''(\mathcal{L}_2)$. ✣

As an immediate consequence in the case where $\mathcal{L}_1$ is delta-normal, we have

COROLLARY 3.8. Let $\mathcal{L}_1 \subset \mathcal{L}_2$. Suppose $\mathcal{L}_1$ is delta-normal and $\mathcal{L}_1$ semi-separates $\mathcal{L}_2$. If $\mu \in M_\sigma(\mathcal{L}_1)$ then $\mu' = \mu''(\mathcal{L}_2)$.

PROOF. By Theorem 3.5(a), if $\mathcal{L}_1$ is delta-normal and if $\mu \in M_\sigma(\mathcal{L}_1)$, then $\mu' = \mu''(\mathcal{L}_1)$ Hence by Theorem 3.8 the corollary follows. ✣

We shall build up in the next section a complete characterization of semi-separation of a pair of lattices $\mathcal{L}_1$, $\mathcal{L}_2$ with $\mathcal{L}_1 \subset \mathcal{L}_2$, in terms of associated outer measures of elements of $M_R(\mathcal{L}_1)$

4. LATTICE SEPARATION

We consider consequences of lattice separation properties between pairs of lattices on some of the associated outer measures introduced earlier Throughout $\mathcal{L}_1$ and $\mathcal{L}_2$ will denote lattices of subsets of X

THEOREM 4.1. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ and suppose $\mathcal{L}_1$ semi-separates $\mathcal{L}_2$ If $\mu \in M_R(\mathcal{L}_1)$ then $\tilde{\mu} = \mu'(\mathcal{L}_2)$.

PROOF. By Theorem 3.3(a) we have $\mu' \leq \tilde{\mu}$, thus $\mu' \leq \tilde{\mu}(\mathcal{L}_2)$. Suppose $\mu' \neq \tilde{\mu}(\mathcal{L}_2)$, then there exists $L_2 \in \mathcal{L}_2$ such that $\mu'(L_2) < \tilde{\mu}(L_2)$ Since

$$\mu'(L_2) = \inf\{\mu(L_1')|L_2 \subset L_1', L_1 \in \mathcal{L}_1\},$$

there exists $L_1 \in \mathcal{L}_1$ such that $L_2 \subset L_1'$ and $\mu(L_1') < \tilde{\mu}(L_2)$. By semi-separation there exists $A_1 \in \mathcal{L}_1$ such that $L_2 \subset A_1 \subset L_1'$ Then

$$\mu(A_1) \leq \mu(L_1') < \tilde{\mu}(L_2).$$

But by monotonicity, $\tilde{\mu}(L_2) \leq \tilde{\mu}(A_1) = \mu(A_1)$ Therefore

$$\tilde{\mu}(L_2) \leq \mu(A_1) \leq \mu(L_1') < \tilde{\mu}(L_2),$$

a contradiction Hence $\tilde{\mu} = \mu'(\mathcal{L}_2)$ ✣

REMARK 1. We note that if for any $\mu \in I_R(\mathcal{L}_1)$ we have $\tilde{\mu} = \mu'(\mathcal{L}_2)$, then $\mathcal{L}_1$semi-separates $\mathcal{L}_2$ (see [6,9]). Theorem 4.1 also appears in [4]; we have included a slightly different proof for completeness We next consider separation between $\mathcal{L}_1$ and $\mathcal{L}_2$

THEOREM 4.2. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ and let $\nu \in M_R(\mathcal{L}_2)$ be an extension of $\mu \in M_R(\mathcal{L}_1)$ If $\mathcal{L}_1$ separates $\mathcal{L}_2$ then ν is $\mathcal{L}_1$-regular on $\mathcal{L}_2$' and $\nu = \tilde{\mu}(\mathcal{L}_2)$.

PROOF. Since $\nu \in M_R(\mathcal{L}_2)$, for $L_2 \in \mathcal{L}_2$

$$\nu(L_2') = \sup\left\{\nu(\tilde{L}_2)|\tilde{L}_2 \subset L_2', \tilde{L}_2 \in \mathcal{L}_2\right\}.$$

Then given $\epsilon > 0$ there exists $\tilde{L}_2 \in \mathcal{L}_2$ such that $\tilde{L}_2 \subset L_2'$ and $\nu(L_2') - \epsilon < \nu(\tilde{L}_2)$ By separation there exist L_1, $\tilde{L}_1 \in \mathcal{L}_1$ such that $\tilde{L}_2 \subset \tilde{L}_1 \subset L_1' \subset L_2'$. Hence

$$\nu(L_2') - \epsilon < \nu(\tilde{L}_1) \leq \nu(L_2').$$

Thus clearly $\nu(L_2') = \sup\left\{\nu(\tilde{A})|\tilde{A} \subset L_2', \tilde{A} \in \mathcal{L}_1\right\}$, i.e., ν is $\mathcal{L}_1$-regular on $\mathcal{L}_2'$

Now by Theorem 3.6, $\nu = \mu'(\mathcal{L}_2)$. But since $\mathcal{L}_1$ separates $\mathcal{L}_2$ and, consequently, semi-separates $\mathcal{L}_2$, Theorem 4 1 yields $\mu' = \tilde{\mu}(\mathcal{L}_2)$. ✣

REMARK 2. Clearly if $\mathcal{L}_1$ separates $\mathcal{L}_2$ then the regular extension $\nu \in M_R(\mathcal{L}_2)$ of $\mu \in M_R(\mathcal{L}_1)$ is unique. It is also to be noted that if we just assume $\mathcal{L}_1 \subset \mathcal{L}_2$, and for any $\mu \in I_R(\mathcal{L}_1)$ and any extension $\nu \in I_R(\mathcal{L}_2)$, ν is $\mathcal{L}_1$-regular on $\mathcal{L}_2'$ and if $\mathcal{L}_1$ semi-separates $\mathcal{L}_2$, then $\mathcal{L}_1$ separates $\mathcal{L}_2$ (see [5]). In summary, $\mathcal{L}_1$ separates $\mathcal{L}_2$ if and only if $\nu = \tilde{\mu}$ on $\mathcal{L}_2$ for any $\mu \in M_R(\mathcal{L}_1)$ and any $\nu \in M_R(\mathcal{L}_2)$ extending μ.

For coseparation of lattices, we can obtain even stronger results.

THEOREM 4.3. Suppose $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mu \in M(\mathcal{L}_1)$ If $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ then there exists a unique $\nu \in M_R(\mathcal{L}_2)$ such that $\mu \leq \nu|(\mathcal{L}_1)$ and $\mu(X) = \nu(X)$.

PROOF. Since $\nu \in M_R(\mathcal{L}_2)$, $\nu = \nu'(\mathcal{L}_2)$ where $\mu \leq \nu(\mathcal{L}_1)$. If $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ coseparates $\mathcal{L}_2$, then $\mathcal{L}_1$ coseparates itself or equivalently, $\mathcal{L}_1$ is normal. Thus by statement (4) in section 2, $(\nu|)' = \mu'(\mathcal{L}_1)$; hence

$$\mu \leq \nu| = (\nu|)' = \mu'(\mathcal{L}_1).$$

Now if $\nu_1 \in M_R(\mathcal{L}_2)$ and if $\mu \leq \nu_1|(\mathcal{L}_1)$ then $\mu \leq \nu_1| = (\nu_1|)' = \mu'(\mathcal{L}_1)$; similarly if $\nu_2 \in M_R(\mathcal{L}_2)$ and if $\mu \leq \nu_2|(\mathcal{L}_1)$ then $\mu \leq \nu_2| = (\nu_2|)' = \mu'(\mathcal{L}_1)$ Hence $\nu_1 = \nu_2$. ✣

Continuing, we have

THEOREM 4.4. Suppose $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ coseparates $\mathcal{L}_2$. Let $\mu \in M(\mathcal{L}_1)$ and $\nu \in M_R(\mathcal{L}_2)$ where $\mu(X) = \nu(X)$. If $\mu \leq \nu|(\mathcal{L}_1)$ then

(a) $\nu(L_2') = \sup\{\mu(L_1)|L_1 \subset L_2', L_1 \in \mathcal{L}_1\}$ for $L_2 \in \mathcal{L}_2$.

(b) $\nu = \mu'(\mathcal{L}_2)$

PROOF. (a) Let $A_2 \in \mathcal{L}_2$, then

$$\nu(A_2) = \inf\{\nu(L_2') | A_2 \subset L_2', L_2 \in \mathcal{L}_2\}$$

since $\nu \in M_R(\mathcal{L}_2)$ Hence given $\epsilon > 0$ there exists $L_2 \in \mathcal{L}_2$ such that $A_2 \subset L_2'$ and $\nu(L_2') - \epsilon < \nu(A_2)$ By coseparation, there exist $A_1, L_1 \in \mathcal{L}_1$ such that $A_2 \subset A_1' \subset L_1 \subset L_2'$ Therefore

$$\mu(L_1) \geq \mu(A_1') \geq \nu(A_1') \geq \nu(A_2) > \nu(L_2') - \epsilon.$$

Hence $\nu(L_2') = \sup\{\mu(L_1) | L_1 \subset L_2', L_1 \in \mathcal{L}_1\}$

(b) Let $L_2 \in \mathcal{L}_2$, then

$$\begin{aligned}\mu'(L_2) &= \inf\{\mu(L_1') | L_2 \subset L_1', L_1 \in \mathcal{L}_1\} \\ &= \inf\{\mu(X) - \mu(L_1) | L_2 \subset L_1', L_1 \in \mathcal{L}_1\} \\ &= \mu(X) - \sup\{\mu(L_1) | L_1 \subset L_1', L_1 \in \mathcal{L}_1\} \\ &= \nu(X) - \sup\{\mu(L_1) | L_1 \subset L_2', L_1 \in \mathcal{L}_1\} \\ &= \nu(X) - \nu(L_2').\end{aligned}$$

Hence $\mu'(L_2) = \nu(L_2)$ and so $\nu = \mu'(\mathcal{L}_2)$. ✥

REMARK 3. Part (b) above gives an alternative proof to Theorem 4 3. Suppose $\nu_1, \nu_2 \in M_R(\mathcal{L}_2)$ such that $\mu \leq \nu_1|(\mathcal{L}_1)$ and $\mu \leq \nu_2|(\mathcal{L}_1)$. Then $\nu_1 = \mu'(\mathcal{L}_2)$ and $\nu_2 = \mu'(\mathcal{L}_2)$ Hence $\nu_1 = \nu_2$.

We can explore Theorem 4.4 further by imposing conditions on the measure μ and the lattice $\mathcal{L}_1$, and see how this affects the measure ν on the larger lattice $\mathcal{L}_2$

THEOREM 4.5. Suppose $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ Let $\mu \in M_\sigma(\mathcal{L}_1)$and $\nu \in M_R(\mathcal{L}_2)$ where $\mu(X) = \nu(X)$

(a) If $\mu \leq \nu|(\mathcal{L}_1)$ then $\nu \in M_\sigma(\mathcal{L}_2')$.

(b) If $\mu \leq \nu|(\mathcal{L}_1)$ and μ'' is regular then $\nu \in M^\circ(\mathcal{L}_2')$, provided $\mathcal{L}_1$ is a delta lattice.

PROOF. (a) Let $B_n \in \mathcal{L}_2$ and $B_n' \downarrow \emptyset$. By Theorem 4.4(a), given $\epsilon > 0$ there exists $A_n \in \mathcal{L}_1$ such that

$$A_n \subset B_n' \quad \text{and} \quad \nu(B_n') - \epsilon < \mu(A_n),$$

and we may assume that $A_n \downarrow \emptyset$ Moreover, $\mu(A_n) \to 0$ since $\mu \in M_\sigma(\mathcal{L}_1)$ Hence, $\lim_{n\to\infty} \nu(B_n') = 0$, i.e , $\nu \in M_\sigma(\mathcal{L}_2')$

(b) Since $\mathcal{L}_1$ is delta-normal, $\mu' = \mu''(\mathcal{L}_1)$ by Theorem 3.5(a), and thus by Corollary 3.8, $\mu' = \mu''(\mathcal{L}_2)$ It follows from Theorem 4.4(b) that $\nu = \mu' = \mu''(\mathcal{L}_2)$. Now let

$$L_n' \downarrow L' \quad \text{where} \quad L, L_n \in \mathcal{L}_2.$$

Then $L_n \uparrow L$ and since μ'' is regular,

$$\mu''(L_n) \uparrow \mu''(L).$$

Since $\nu = \mu''(\mathcal{L})$, $\nu(L_n) \uparrow \nu(L)$; therefore

$$\nu(L_n') \downarrow \nu(L').$$

Hence $\nu \in M^\circ(\mathcal{L}_2')$. ✥

So far we have concentrated on ν as a regular measure "enlargement" of $\mu \in M(\mathcal{L}_1)$ from $\mathcal{A}(\mathcal{L}_1)$ to $\mathcal{A}(\mathcal{L}_2)$ on coseparation lattices. We now turn our attention to measure extensions

THEOREM 4.6. Suppose $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ Let $\mu \in M(\mathcal{L}_1)$ and $\nu \in M_R(\mathcal{L}_2)$ where $\mu(X) = \nu(X)$. If $\mu \leq \nu|(\mathcal{L}_1)$ then

(a) $\tilde{\mu} \leq \nu = \mu'(\mathcal{L}_2)$

(b) and if $\tilde{\mu} = \nu = \mu'(\mathcal{L}_2)$, then $\mu \in M_R(\mathcal{L}_1)$ and $\mu = \nu|(\mathcal{L}_1)$.

PROOF. (a) By Theorem 4 4(b), $\nu = \mu'(\mathcal{L}_2)$ Let $L_2 \in \mathcal{L}_2$, then

$$\nu(L_2) = \mu'(L_2) = \inf\{\mu(L_1') | L_2 \subset L_1', L_1 \in \mathcal{L}_1\}.$$

Hence given $\epsilon > 0$ there exists $L_1 \in \mathcal{L}_1$ such that $L_2 \subset L_1'$ and $\mu(L_1') - \epsilon \leq \nu(L_2)$. By semi-separation, there exists $A_1 \in \mathcal{L}_1$ such that $L_2 \subset A_1 \subset L_1'$ Hence

$$\tilde{\mu}(L_2) \leq \tilde{\mu}(A_1) = \mu(A_1) \leq \mu(L_1') \leq \nu(L_2) + \epsilon,$$

and thus $\tilde{\mu}(L_2) \leq \nu(L_2)$. Therefore $\tilde{\mu} \leq \nu = \mu'(\mathcal{L}_2)$

(b) If $\tilde{\mu} = \mu' = \nu(\mathcal{L}_2)$, then $\tilde{\mu} = \mu' = \nu(\mathcal{L}_1)$ since $\mathcal{L}_1 \subset \mathcal{L}_2$. But $\tilde{\mu} = \mu(\mathcal{L}_1)$, hence $\mu = \mu'(\mathcal{L}_1)$ and equivalently, $\mu \in M_R(\mathcal{L}_1)$. ✥

In the course of developing our measure extension and enlargement results, we have thus extended previously known results such as those on coallocation lattices [3]. Finally, we note some consequences of assuming the $\mathcal{L}_1$ lattice to be delta or almost countably compact

THEOREM 4.7. Let $\mathcal{L}_1 \subset \mathcal{L}_2$. Then

(a) If $\mathcal{L}_1$ is a delta lattice and $\mathcal{L}_1$ coseparates $\mathcal{L}_2$, and if $\mu \in M_\sigma(\mathcal{L}_1)$ then $\mu' = \mu''(\mathcal{L}_2)$.

(b) If $\mathcal{L}_1$ is almost countably compact and $\mathcal{L}_1$ coseparates $\mathcal{L}_2$, and if $\nu \in M_R(\mathcal{L}_2)$ then $\nu \in M_\sigma(\mathcal{L}_2')$.

PROOF. (a) This is just a special case of Corollary 3.8, since if $\mathcal{L}_1 \subset \mathcal{L}_2$ and if $\mathcal{L}_1$ coseparates $\mathcal{L}_2$, then $\mathcal{L}_1$ coseparates itself or equivalently, $\mathcal{L}_1$ is normal.

(b) Consider $\nu| \in M_R(\mathcal{L}_1)$. Then

$$\nu| \leq \mu(\mathcal{L}_1'), \quad \mu \in M_R(\mathcal{L}_1') \quad \text{and} \quad \mu(X) = \nu(X).$$

Hence $\mu \leq \nu|(\mathcal{L}_1)$ and $\mu \in M_\sigma(\mathcal{L}_1)$ since $\mathcal{L}_1$ is almost countably compact (see section 2). From Theorem 4.5(a), it follows that $\nu \in M_\sigma(\mathcal{L}_2')$. ✥

We close this section with several extensions and ramifications of results found in [4].

THEOREM 4.8. Let $\mathcal{L}$ be normal and $\mu \in I_\sigma(\mathcal{L})$. Suppose

$$A = \bigcap_{n=1}^{\infty} B_n' \quad \text{where} \quad A \in \mathcal{L} \quad \text{and} \quad B_n \in \mathcal{L} \quad \text{for all} \quad n.$$

Then $A \in S_{\mu''}$.

PROOF. Recall that for $\mu \in I_\sigma(\mathcal{L})$,

$$S_{\mu''} = \left\{ E \subset X | E \supset \bigcap_{n=1}^{\infty} L_n \text{ or } E' \supset \bigcap_{n=1}^{\infty} L_n,\ L_n \in \mathcal{L},\ \mu(L_n) = 1 \text{ for all } n \right\}, \quad \text{(see [4])}.$$

Case 1. If $\mu''(A) = 0$ then $A \in S_{\mu''}$

Case 2. If $\mu''(A) = 1$ then since $\mathcal{L}$ is normal, there exist C_n, $D_n \in \mathcal{L}$ such that

$$A \subset C_n' \subset D_n \subset B_n'.$$

Therefore $\mu''(C_n') = 1$, so $\mu'(C_n') = \mu(C_n') = 1$. Hence

$$\mu(D_n) = 1 \quad \text{for all} \quad n, \quad \text{where} \quad A = \bigcap_{n=1}^{\infty} D_n,$$

thus $A \in S_{\mu''}$. ✥

THEOREM 4.9. Let $\mathcal{L}$ be normal and $\mu \in M^\circ(\mathcal{L})$. Suppose

$$A = \bigcap_{n=1}^{\infty} B_n' \quad \text{where} \quad A \in \mathcal{L} \quad \text{and} \quad B_n \in \mathcal{L} \quad \text{for all} \quad n.$$

Then $A \in S_{\mu'}$.

PROOF. By normality there exist C_n, $D_n \in \mathcal{L}$ such that $A \subset C_n' \subset D_n \subset B_n'$. Then

$$A = \cap C'_n = \cap D_n = \cap B'_n.$$

We may assume $C'_n \downarrow$ and $D_n \downarrow$ Then

$$\mu(A) = \lim_{n\to\infty} \mu(D_n)$$

since $\mu \in M^\circ(\mathcal{L})$ Hence

$$\mu(A) \leq \mu(C'_n) \leq \mu(D_n) \to \mu(A).$$

Thus

$$\mu(A) = \lim_{n\to\infty} \mu(C'_n) \geq \mu'(A).$$

But in general $\mu \leq \mu'(\mathcal{L})$, therefore

$$\mu(A) = \mu'(A).$$

It follows from Theorem 3 1(4) that

$$A \in S_{\mu'} \cap \mathcal{L} = \{L \in \mathcal{L} | \mu'(L) = \mu(L)\}. \quad \blacklozenge$$

As special cases of the above theorem, we have the following.

COROLLARY 4.10. Suppose $\mathcal{L}$ is a normal and complement generated lattice If $\mu \in M^\circ(\mathcal{L})$ then $\mathcal{L} \subset S'_{\mu'}$ and $\mu \in M^\sigma_R(\mathcal{L})$.

PROOF. If $\mathcal{L}$ is complement generated then every set $A \in \mathcal{L}$ can be written as $A = \bigcap_{n=1}^{\infty} B'_n$, $B_n \in \mathcal{L}$. It follows from the above theorem that

$$\mathcal{L} \subset S_{\mu'} \cap \mathcal{L} = \{L \in \mathcal{L} | \mu'(L) = \mu(L)\}.$$

Hence $\mu = \mu'(\mathcal{L})$ or equivalently $\mu \in M_R(\mathcal{L})$, and since $\mu \in M^\circ(\mathcal{L})$ we have $\mu \in M^\sigma_R(\mathcal{L})$ $\blacklozenge$

COROLLARY 4.11. Let $\mathcal{L}$ be normal and $\mu \in \Gamma^\circ(\mathcal{L})$ Suppose

$$A = \bigcap_{n=1}^{\infty} B'_n \quad \text{where} \quad A \in \mathcal{L},\ B_n \in \mathcal{L} \quad \text{for all} \quad n.$$

Then $A \in S_{\mu'}$ and thus $A \in S_{\mu''}$

PROOF. It is a special case of Theorems 3.4(d) and 4.9. $\blacklozenge$

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