ON THE EXTREME POINTS OF SOME CLASSES OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. Let $U$ be the unit disk, $D \supset U$ an open connected set and $z_0 \in D$. Let also $P(z_0, c, D)$ be the class of holomorphic functions in $D$ for which $f(z_0) = c$ and $\text{Re} f(z) > 0$ in $U$. We find the extreme points of the class $P(z_0, c, D)$.

KEY WORDS AND PHRASES. Extreme points, positive real part.

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1. INTRODUCTION.

Let $U$ be the unit disk $\{z : |z| < 1\}$, $D \supset U$ an open connected set, $z_0 \in D$ and $H(D)$ be the class of holomorphic functions in $D$. By $P(z_0, c, D)$ we denote the class of the functions $f \in H(D)$ for which $f(z_0) = c$ and $\text{Re} f(z) > 0$ in $U$. Let $EP(z_0, c, D)$ be the subclass of the extreme points of the above class for $P = P(0, 1, U)$ it has proven [1] that 

$$EP = \{(e + z)(e - z)^{-1} : e \in \partial U - D\},$$

In this paper we find the points of the subclass $EP(z_0, c, D)$.

2. MAIN RESULT.

THEOREM. (i) If $(1 - |z_0|)\text{Re} c \leq 0$ then $EP(z_0, c, D) = \emptyset$. (ii) If $(1 - |z_0|)\text{Re} c > 0$ then $f \in EP(z_0, c, D)$ iff it has the form

$$f(z) = x_1(\frac{\epsilon + z}{\epsilon - z}) + ix_2,$$

where $\epsilon \in \partial U - D$, $x_1 = \text{Re} \{\text{Re}(\frac{\epsilon + z}{\epsilon - z_0})\}^{-1}$ and $x_2 = \text{Im} c - x_1, \text{Im}(\frac{\epsilon + z}{\epsilon - z_0})$.

PROOF. Let $f \in P(z_0, c, D)$ with $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ in $U$. Let also $r < 1$, $S$ be a complex number and $M > 0$ such that $0 < 2|S| < M$ and $z \in \partial U$. Since

$$[1 \pm \frac{1}{M}(Sz + \overline{S}z^{-1})]\text{Re} f(rz) > 0$$

then

$$\text{Re} [f(rz) \pm \frac{1}{M}(Sz f(rz) + \overline{S} \sum_{n=0}^{\infty} \alpha_n r^n z^{n-1} + S\overline{\alpha}_0 z)]$$

(1)

By the maximum principle for harmonic functions it follows that (1) holds for every $z \in U$. Therefore for $r \to 1$ we have $\text{Re} (f(z) \pm u_1(z)) > 0$ in $U$ where

$$u_1(z) = \frac{1}{M}[^{\overline{S}}z^{-1}(f(z) - \alpha_0) + S\overline{\alpha}_0 z + Sz f(z)]$$

(2)
Choosing appropriate \( S \neq 0 \) we get \( \text{Re}u_1(z_0) = 0 \). Setting \( u(z) = u_1(z) - i \text{Im}u_1(z_0) \) from \( u(z_0) = 0 \) it follows that \( f \pm u \in \mathcal{P}(z_0, c, D) \).

Let now \( f \in \mathcal{E}P(z_0, c, D) \). Then it is obvious that \( u(z) = 0 \) in \( D \). If we set \( S = |S|e^{i(\psi + \delta)} \) then from equality \( u = 0 \) we conclude that \( f \) has the form

\[
f(z) = \frac{\xi_1(1 + z^2e^{2i\psi}) + \xi_2ze^{i\psi}}{1 - z^2e^{2i\psi}} + i\xi_3 =
\]

\[
\frac{1}{2}(\xi_1 + \xi_2)\left(1 + \frac{e^{i\psi}z}{1 - e^{i\psi}z}\right) + \frac{1}{2}(\xi_1 - \xi_2)\left(1 - \frac{e^{i\psi}z}{1 + e^{i\psi}z}\right) + i\xi_3,
\]

where \( \xi_1, \xi_2, \xi_3 \in \mathbb{R} \).

We now prove that \( |\xi_2| = 2\xi_1 \). From the Caratheodory’s inequality we have \( |f'(0)| \leq 2\text{Re}f(0) \) and hence \( |\xi_2| \leq 2\xi_1 \). If \( |\xi_2| < 2\xi_1 \) then there are \( \xi_1^*, \xi_2^* \) such that \( 0 < |\xi_1^*| < \xi_1 + \frac{\xi_2}{2}, 0 < |\xi_2^*| < \xi_1 - \frac{\xi_2}{2} \), and \( \text{Re}u_1^*(z_0) = 0 \), where

\[
u_1^*(z) = \xi_1^*(1 + \frac{e^{i\psi}z}{1 - e^{i\psi}z}) + \xi_2^*(1 - \frac{e^{i\psi}z}{1 + e^{i\psi}z})
\]

Setting \( u^*(z) = u_1^*(z) - i\text{Im}u_1^*(z_0) \) then \( f \pm u^* \in \mathcal{P}(z_0, c, D) \). Since \( f \in \mathcal{E}P(z_0, c, D) \) it follows that \( u^* = 0 \) and hence \( \xi_1^* = \xi_2^* = 0 \). Therefore if \( f \in \mathcal{E}P(z_0, c, D) \) then \( |\xi_2| = 2\xi_1 \) and hence \( f \) has the form

\[
f(z) = z_1\left(\frac{\epsilon + z}{\epsilon - z}\right) + i\epsilon \epsilon_1, z_1 > 0, \epsilon \epsilon_2 > 0, \epsilon \epsilon \in \partial U - D.
\]

From (4) we have

\[
x_1 = \text{Re}[\text{Re}(\frac{\epsilon + z_0}{\epsilon - z_0})]^{-1} > 0 \text{ and hence } (1 - |z_0|)\text{Re} > 0.
\]

Let \( f \in \mathcal{P}(z_0, c, D) \) and having the form (4). Let also \( 0 < \lambda < 1 \) and \( f_1, f_2 \in \mathcal{E}P(z_0, c, D) \) such that \( f = \lambda f_1(1 - \lambda)f_2 \). Then

\[
\frac{\epsilon + z}{\epsilon - z} = \lambda^*g_1(z) + (1 - \lambda^*)g_2(z) \quad \text{in } U,
\]

where

\[
\lambda^* = \lambda \frac{\text{Re}f_1(0)}{\text{Re}f(0)}, \quad g_i(z) = \frac{f_i(z) - i\text{Im}f_i(0)}{\text{Re}f_i(0)}, \quad i = 1, 2.
\]

Since

\[
\frac{\epsilon + z}{\epsilon - z} \in \mathcal{E}P \text{ and } g_i \in \mathcal{P}
\]

then

\[
\frac{\epsilon + z}{\epsilon - z} = g_1(z) = g_2(z) \text{ in } U.
\]

From the identity Theorem and the restrictions \( f(z_0) = f_1(z_0) = f_2(z_0) = c \), we obtain \( f = f_2 \) and hence \( f \in \mathcal{E}P(z_0, c, D) \).

REFERENCES

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