SUBCLASSES OF UNIVALENT FUNCTIONS SUBORDINATE TO CONVEX FUNCTIONS

YONG CHAN KIM
Department of Mathematics
Yungnam University
Gyongsan 712-749, KOREA

IL BONG JUNG
Department of Mathematics
Kyungpook National University
Taegu 702-701, KOREA

(Received May 17, 1995 and in revised form August 30, 1995)

ABSTRACT. In this paper, we define a new subclass $M_\alpha(A, B)$ of univalent functions and investigate several interesting characterization theorems involving a general class $S^*[A, B]$ of starlike functions.

KEY WORDS AND PHRASES: Univalent function, subordination, $\alpha$-convex function

1991 AMS SUBJECT CLASSIFICATION CODES: 30C45, 30D30

1. INTRODUCTION AND DEFINITIONS

Let $A$ denote the class of functions normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, let $S$ denote the class of all functions in $U$ which are univalent in $U$.

A function $f(z)$ belonging to $S$ is said to be starlike of order $\alpha$ ($0 \leq \alpha < 1$) if and only if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in U; 0 \leq \alpha < 1).$$

We denote by $S^*(\alpha)$ the subclass of $S$ consisting of functions which are starlike of order $\alpha$.

A function $f(z)$ belonging to $S$ is said to be convex of order $\alpha$ ($0 < \alpha < 1$) if and only if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in U; 0 < \alpha < 1).$$

We denote by $K(\alpha)$ the subclass of $S$ consisting of functions which are convex of order $\alpha$.

We note that $S^*(\alpha) \subseteq S^*(0) \equiv S^* (0 \leq \alpha < 1)$

and

$$K(\alpha) \subseteq K(0) \equiv K (0 \leq \alpha < 1).$$

With a view to introducing an interesting family of analytic functions, we should recall the concept of subordination between analytic functions. Given two functions $f(z)$ and $g(z)$, which are analytic in $U$, the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $h(z)$, analytic in $U$ with

$$h(0) = 0 \quad \text{and} \quad |h(z)| < 1,$$

such that

$$f(z) = g(h(z)) \quad (z \in U).$$

We denote this subordination by

$$f(z) \prec g(z).$$
In particular, if \( g(z) \) is univalent in \( U \), the subordination (18) is equivalent to
\[
f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]

Janowski [1] introduced the class \( \mathcal{P}[A, B] \). For \( -1 < B < A < 1 \), a function \( p \), analytic in \( U \) with \( p(0) = 1 \), belongs to the class \( \mathcal{P}[A, B] \) if \( p(z) \) is subordinate to \( (1 + Az)/(1 + Bz) \). Also \( S^*[A, B] \) and \( K[A, B] \) denote the subclasses of \( S \) consisting of all functions \( f(z) \) such that
\[
\frac{zf''(z)}{f'(z)} \in \mathcal{P}[A, B] \quad \text{and} \quad \left( \frac{zf'(z)}{f(z)} \right)' \in \mathcal{P}[A, B],
\]
respectively. We note that \( S^*[-1, 1] = S^* \) and \( K[-1, 1] = K \).

**Definition 1.** Let \( \alpha \) be a real number. A function \( f(z) \) belonging to the class \( A \) with \( (f(z)/z)f'(z) \neq 0 \) is said to be \( \alpha \)-convex in \( U \) if and only if
\[
\Re \left[ (1 - \alpha) \frac{zf''(z)}{f'(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > 0.
\]

Also we denote the class of \( \alpha \)-convex functions by \( M_\alpha \). Then it is easy to see that
\[
M_\alpha = \left\{ f \in S : \Re \left[ (1 - \alpha) \frac{zf''(z)}{f'(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > 0, \quad z \in U \right\}.
\]

See Eenigenberg and Miller [5] for further information on them.

We now define the class \( M_\alpha(A, B) \) as follows: If \( \alpha \) is a real number, then
\[
M_\alpha(A, B) = \left\{ f \in S : \left[ (1 - \alpha) \frac{zf''(z)}{f'(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \left( \frac{1 + Az}{1 + Bz} \right) > -1 < B < A < 1, \quad z \in U \right\}.
\]

Clearly, we have
\[
M_\alpha(1, -1) = M_\alpha, \quad M_{1}(A, B) = K[A, B],
\]
and
\[
M_0(A, B) = S^*[A, B].
\]

**2. MAIN RESULTS**

Applying the method of the integral representation [2] for functions in \( M_\alpha(A, B) (\alpha > 0) \), it is not difficult to deduce

**Lemma 1.** The function \( f(z) \) is in \( M_\alpha(A, B), \alpha > 0 \), if and only if there exists a function \( g(z) \) belonging to the class \( S^*[A, B] \) such that
\[
f(z) = \left[ \frac{1}{\alpha} \int_0^z (g(t))^{1/\alpha} t^{-1} dt \right]^\alpha.
\]

**Proof.** Setting \( g(z) = f(z) \left( \frac{zf''(z)}{f'(z)} \right)^\alpha \), so that (2.1) is satisfied, we observe that
\[
\frac{zf''(z)}{g(z)} = (1 - \alpha) \frac{zf''(z)}{f'(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right).
\]
Hence \( f \in M_\alpha(A, B) \) if and only if \( g \in S^*[A, B] \).

Before stating our first theorem, we need the following definition

**Definition 2.** Let \( c \) be a complex number such that \( \Re c > 0 \), and let
If $h$ is the univalent function $h(z) = 2Nz/(1 - z^2)$ and $b = h^{-1}(c)$, then we define the "open door" (cf [3]) function $Q_c$ as
\[ Q_c(z) = h[(z + b)/(1 + bz)], \quad z \in \mathcal{U}. \] (2.3)

**THEOREM 1.** Let $f \in \mathcal{M}_b (A, B) (\alpha > 0)$, and let
\[ \left( \frac{1 + Az}{1 + bz} \right) < \alpha Q_b(z). \] (2.4)

Then $f \in S^*$

**PROOF.** Since $f \in \mathcal{M}_b (A, B) (\alpha > 0)$, it follows that there exists a function $g \in S^* [A, B]$ such that
\[ f(z) = \left[ \frac{1}{\alpha} \int_0^z \{ g(t) \}^{1/\alpha} t^{-1} dt \right]^\alpha, \] (2.5)
by using Lemma 1. By the hypothesis, we also have
\[ \frac{1}{\alpha} \left( \frac{zg'(z)}{g(z)} \right) - \frac{1}{\alpha} \left( \frac{1 + Az}{1 + Bz} \right) < Q_b(z). \] (2.6)

Thus, by a result of Miller and Mocanu ([3], Corollary 3.1), we have
\[ f(z) = \left[ \frac{1}{\alpha} \int_0^z \{ g(t) \}^{1/\alpha} t^{-1} dt \right]^\alpha \in S^*. \]

**LEMMA 2.** (Mocanu [4]) Let $\mathcal{P}$ be an analytic function in $\mathcal{U}$ satisfying $\mathcal{P} \prec Q_c$. If $p$ is analytic in $\mathcal{U}$, $p(0) = 1/c$, and
\[ zp'(z) + \mathcal{P}(z)p(z) = 1, \] (2.7)
then $\text{Re} \ p(z) > 0$ in $\mathcal{U}$

Making use of Lemma 2, we now prove

**THEOREM 2.** Let $f \in \mathcal{M}_b (A, B) (\alpha > 0)$, and let
\[ \frac{zf'(z)}{f(z)} + \frac{f(z)}{zf'(z)} - 1 \prec Q_1. \] (2.8)

Then $f \in S^*[A, B]$.

**PROOF.** If we set $p(z) = zf'(z)/f(z)$, then
\[ p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}. \] (2.9)

Hence
\[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) = p(z) + \alpha \frac{zp'(z)}{p(z)}. \] (2.10)

Since $f \in \mathcal{M}_b (A, B)$,
\[ p(z) + \alpha \frac{zp'(z)}{p(z)} \prec \frac{1 + Az}{1 + Bz}. \] (2.11)

Setting $\mathcal{P}(z) = p(z) + 1/p(z) - 1$, we obtain
\[ zp'(z) + \mathcal{P}(z)p(z) = 1 \] (2.12)
and \( \mathcal{P} \prec Q_1 \) by the hypothesis (2.8).

Thus, by Lemma 2, we have

\[
\text{Re } p(z) > 0 \quad (z \in \mathcal{U}).
\]  

(2.13)

Since \( \alpha > 0 \),

\[
\text{Re} \left\{ \frac{1}{\alpha} p(z) \right\} > 0 \quad (z \in \mathcal{U}).
\]  

(2.14)

Also \( (1 + Az)/(1 + Bz) \) (with \( -1 \leq B < A \leq 1 \)) is a convex univalent function. Therefore, by appealing to a known result ([6], Theorem 7), we conclude from (2.11) and (2.14) that

\[
p(z) \lesssim \frac{1 + Az}{1 + Bz}.
\]  

(2.15)

This evidently completes the proof of Theorem 2.

As an example of ([7], Corollary 3.2, see also [9]), consider the case when \( \alpha > 0 \), \( -1 \leq B < A \leq 1 \), and \( A \neq B \). Then the differential equation

\[
q(z) + \alpha \frac{zq'(z)}{q(z)} = \frac{1 + Az}{1 + Bz}
\]  

(2.16)

has a univalent solution given by

\[
q(z) = \begin{cases} 
\frac{z^{\frac{1}{\alpha}} (1 + Bz)^{\frac{1}{\alpha}} (A^\beta \sqrt[A]{B})}{\frac{1}{\alpha} \int_0^1 t^{\beta - 1} (1 + Bt)^{\frac{1}{\alpha}} (A^\beta \sqrt[A]{B}) dt} & \text{if } B \neq 0 \\
\frac{z^\frac{1}{\alpha} e^{\beta z}}{\frac{1}{\alpha} \int_0^1 t^{\beta - 1} e^{\beta t} dt} & \text{if } B = 0.
\end{cases}
\]  

(2.17)

If \( p(z) \) is analytic in \( \mathcal{U} \) and satisfies

\[
p(z) + \alpha \frac{zp'(z)}{p(z)} \lesssim \frac{1 + Az}{1 + Bz},
\]  

(2.18)

then

\[
p(z) \lesssim q(z) \lesssim \frac{1 + Az}{1 + Bz}.
\]  

(2.19)

Hence, by the equations (2.11) and (2.19), we obtain

**THEOREM 3.** Let \( \alpha > 0 \) and \( f \in \mathcal{M}_\alpha (A, B) \). Then

\[
\frac{zf'(z)}{f(z)} \lesssim q(z) \lesssim \frac{1 + Az}{1 + Bz},
\]  

(2.20)

where \( q(z) \) is given by (2.17).

**THEOREM 4.** \( \mathcal{K}(\alpha) \subset \mathcal{M}_\alpha (1 - 2\alpha, -1) \) (\( 0 \leq \alpha < 1 \)).

**PROOF.** If we define

\[
h_\alpha(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1),
\]  

(2.21)

then we can easily see that \( f \in \mathcal{K}(\alpha) \) if and only if

\[
1 + \frac{zf''(z)}{f'(z)} \lesssim h_\alpha(z)
\]  

(2.22)

(cf. [10], Equation (9)). Hence, by Theorem 1 of [10], we have
Therefore we conclude from [8, Lemma 2.2] that

\[
\left(1 - \alpha \right) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) < h_\alpha(z).
\]  

(2.24)

This completes the proof of Theorem 4

ACKNOWLEDGMENT. This work was partially supported by KOSEF (project No 94-1400-02-01-3) and TGRC-KOSEF, and by the Basic Science Research Institute Program (BSRI-95-1401)

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