SOME PROPERTIES AND CHARACTERIZATIONS OF A -NORMAL FUNCTIONS

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ABSTRACT. Let M be the set of all functions meromorphic on $D = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in (0, 1]$, a function $f \in M$ is called a -normal function of bounded (vanishing) type or $f \in N^a$ (N_0^a), if $\sup_{z \in D} (1 - |z|)^a f^\#(z) < \infty$ ($\lim_{|z| \rightarrow 1} (1 - |z|)^a f^\#(z) = 0$). In this paper we not only show the discontinuity of N^a and N_0^a relative to containment as a varies, which shows $\bigcup_{0 < a < 1} N^a \subset UBC_0$, but also give several characterizations of N^a and N_0^a which are real extensions for characterizations of N and N_0 .

KEY WORDS AND PHRASES: A -normal function, UBC_0 .

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1. INTRODUCTION.

Throughout this paper, let $D = \{z : |z| < 1\}$ be the unit disk in $\mathbb{C}$ and $dm(z)$ the two-dimensional Lebesgue measure on D . Also let $\phi_w(z) = (w - z)/(1 - \bar{w}z)$ to be a canonical Möbius map of D onto D determined by $w \in D$, and let $D(w, r) = \{z \in D : |\phi_w(z)| < r\}$ a pseudohyperbolic disk with center $w \in D$ and radius $r \in [0, 1]$. Suppose that $g(z, w) = -\log|\phi_w(z)|$ is the Green function of D with logarithmic singularity at $w \in D$. Also assume that $a \in (0, 1]$ and M is the class of functions meromorphic on D . For $f \in M$, let $f^\#(z) = |f'(z)|/(1 + |f(z)|^2)$, which is the spherical derivative of f . Further we say f is an a -normal function of bounded type if

$$\|f\|_{N^a} = \sup_{z \in D} (1 - |z|)^a f^\#(z) < \infty, \quad (1.1)$$

and f is an a -normal function of vanishing type if

$$\lim_{|z| \rightarrow 1} (1 - |z|)^a f^\#(z) = 0. \quad (1.2)$$

The families of all a -normal functions of bounded and vanishing type are denoted by N^a and N_0^a , respectively. It is easy to observe that $N_0^a \subset N^a$ and that for $a \in (0, 1)$, N_0^a and N^a are proper subsets of N and N_0 , which are the classical sets of normal and little normal functions, namely, $N = N^1$ and $N_0 = N_0^1$, respectively.

There has been much interesting research on N and N_0 (see [1-3]), and hence we look for N^a and N_0^a to have some analogous properties. In this paper, we first consider the continuity

of the families N^a and N_0^a as a varies, and we find that both are discontinuous, moreover that $\bigcup_{0 < a < 1/2} N^a$ and $\bigcup_{1/2 \leq a < 1} N^a$ are proper subsets of $D_\#$ and UBC_0 , respectively, where $D_\#$ is the family of functions $f \in M$ satisfying

$$\|f\|_{D_\#}^2 = \int_D [f^\#(z)]^2 dm(z) < \infty, \quad (1.3)$$

and UBC_0 is the family of functions $f \in M$ satisfying

$$\lim_{|w| \rightarrow 1} \int_D [f^\#(z)]^2 g(z, w) dm(z) = 0. \quad (1.4)$$

Here, it is worth while mentioning that $D_\# \subset UBC_0$ and that UBC_0 is an important meromorphic counterpart of $VMOA$ —the space of analytic functions with vanishing mean oscillation on D (see [4,8]). We then characterize functions in N^a and N_0^a and obtain three criterions which are extensions of criteria for N and N_0 .

2. CONTINUITY OF N^a AND N_0^a .

In this section, we pay attention to the continuity of N^a and N_0^a . Firstly, we see the monotonicity of N^a and N_0^a . More precisely we have

THEOREM 2.1 Let $a_1, a_2 \in (0, 1]$. If $a_1 < a_2$ then

(i). $N^{a_1} \subset N^{a_2}$.

(ii). $N_0^{a_1} \subset N_0^{a_2}$.

PROOF. It suffices to prove $N^{a_1} \subset N_0^{a_2}$ for $a_1 < a_2$. Let $f \in N^{a_1}$, then $\|f\|_{N^{a_1}} < \infty$ and

$$(1 - |z|)^{a_2} f^\#(z) \leq (1 - |z|)^{a_2 - a_1} \|f\|_{N^{a_1}}.$$

This gives $f \in N_0^{a_2}$, i.e., $N^{a_1} \subseteq N_0^{a_2}$. As to the strict inclusion, we take a function $f_1(z) = (1 - z)^{1 - a_3}$, $a_3 \in (a_1, a_2)$. A simple computation just gives $f_1 \in N_0^{a_2} \setminus N^{a_1}$. In fact,

$$\|f_1\|_{N^{a_1}} = (1 - a_3) \sup_{z \in D} \frac{(1 - |z|)^{a_1} |1 - z|^{-a_3}}{1 + |1 - z|^{2(1 - a_3)}} = \infty.$$

At the same time,

$$\lim_{|z| \rightarrow 1} (1 - |z|)^{a_2} f_1^\#(z) = (1 - a_3) \lim_{|z| \rightarrow 1} \frac{(1 - |z|)^{a_2} |1 - z|^{-a_3}}{1 + |1 - z|^{2(1 - a_3)}} = 0.$$

Thus, $f_1 \in N_0^{a_2} \setminus N^{a_1}$. So, $N^{a_1} \subset N_0^{a_2}$.

Denote by $D_\#^\infty$ the class of functions $f \in M$ with

$$\|f\|_{D_\#^\infty} = \sup_{z \in D} f^\#(z) < \infty. \quad (2.1)$$

For $a \in (0, 1)$, it is easy to see that $D_\#^\infty \subset N_0^a$. Furthermore, Theorem 2.1, together with $N^a \subset N$, $N_0^a \subset N_0$ and [3,4] suggest that we consider the continuity of N^a and N_0^a . For this purpose, we need a corollary which can be viewed as an application of Theorem 2.1.

COROLLARY 2.2 Let $a, b \in (0, 1]$. Then

(i). $\bigcup_{a < b} N_0^a = \bigcup_{a < b} N^a$.

(ii). $\bigcap_{a < b} N_0^b = \bigcap_{a < b} N^b$.

PROOF. (i). On the one hand, the relation: $\bigcup_{a < b} N_0^a \subseteq \bigcup_{a < b} N^a$ is clear. On the other hand, if $f \in \bigcup_{a < b} N^a$, then f must be in some N^a , saying, N^a , where $a \in (0, b)$. However, for $a' \in (a, b)$ we have $f \in N_0^{a'}$ by the proof of Theorem 2.1. So $f \in \bigcup_{a < b} N_0^a$, and hence $\bigcup_{a < b} N_0^a = \bigcup_{a < b} N^a$.

(ii). This part can be proved similarly.

Now, we can state the discontinuity of N^a and of N_0^a .

THEOREM 2.3 Let $a, b \in (0, 1]$. Then

- (i). $\bigcup_{a < b} N^a \subset N_0^b$.
- (ii). $\bigcap_{a < b} N_0^b \supset N^a$.

PROOF. Owing to Theorem 2.1 and Corollary 2.2, we only require proving (1). $\bigcup_{a < b} N_0^a \neq N_0^b$ and (2). $\bigcap_{a < b} N^b \neq N^a$.

First, let us consider (1). If $f_2(z) = \sum_{k=1}^{\infty} \frac{z^{2^k}}{k 2^{k(1-b)}}$, then we get that f_2 is bounded on D . Since $\lim_{k \rightarrow \infty} \frac{2^{k(b-a)}}{k} = \infty$ for $b > a$, $f_2 \notin \bigcup_{a < b} N_0^a$ from [7, Theorem 1]. But it follows that $f_2 \in N_0^b$ again from [7, Theorem 1]. Note that we have here used a fact: $f^\#$ is equivalent to $|f'|$ once f is bounded and analytic on D . The above facts tell us that $\bigcup_{a < b} N_0^a \neq N_0^b$ is true.

Second, let us consider (2). For this, we pick $f_3(z) = \sum_{k=1}^{\infty} \frac{k z^{2^k}}{2^{k(1-a)}}$. It is clear that f_3 is bounded on D . Moreover $f_3 \notin N^a$ by using [7, Theorem 1]. However, $\lim_{k \rightarrow \infty} \frac{k}{2^{k(b-a)}} = 0$ as $b > a$, and then $f_3 \in \bigcap_{a < b} N^b$. That is to say, $\bigcap_{a < b} N^b \neq N^a$.

This completes the proof.

Finally, we discuss a special case of Theorem 2.3. Theorem 2.3 implies that $\bigcup_{0 < a < 1} N^a \subset N_0$. Noting the inclusion: $UBC_0 \subset N_0$ [8], we will naturally ask what is connection between $\bigcup_{0 < a < 1} N^a$ and UBC_0 . It is a little bit surprising to us that $\bigcup_{0 < a < 1} N^a$ is a proper subset of UBC_0 . This result shows that there is a big gap from $\bigcup_{0 < a < 1} N^a$ or $\bigcup_{0 < a < 1} N_0^a$ to N^a or N_0^a . Exactly speaking, we obtain

THEOREM 2.4

- (i). $\bigcup_{0 < a < 1/2} N^a \subset D_\#$.
- (ii). $\bigcup_{1/2 \leq a < 1} N^a \not\subset D_\# \not\subset \bigcup_{1/2 \leq a < 1} N^a \subset UBC_0$.
- (iii). $\bigcap_{0 < a < 1} N_0^a \supset D_\#^\infty$.

PROOF. First of all, setting $f_4(z) = \log(1-z)$, we check that $f_4 \in D_\# \setminus \bigcup_{0 < a < 1} N^a$. Indeed, we have

$$\begin{aligned} \|f_4\|_{D_\#}^2 &= \int_D \frac{1}{|1-z|^2 [1 + |\log(1-z)|^2]^2} dm(z) \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \frac{1}{t[1 + \theta^2 + \log^2 t]^2} dt d\theta \\ &\leq 2\pi \arctan \frac{\pi}{2} \end{aligned} \quad (2.2)$$

and

$$\|f_4\|_{N^a} \geq \lim_{t \rightarrow 1} \frac{(1-t)^{a-1}}{1 + \log^2(1-t)} = \infty \quad (2.3)$$

for any $a \in (0, 1)$.

Now we turn to the proofs of (i), (ii) and (iii).

- (i). If $f \in \bigcup_{0 < a < 1/2} N^a$, then there is an $a \in (0, 1/2)$ such that $f \in N^a$ and thus

$$\|f\|_{D_\#}^2 \leq \|f\|_{N^a}^2 \int_D \frac{1}{(1-|z|)^{2a}} dm(z). \quad (2.4)$$

(2.3) and (2.4) imply that (i) is true.

(ii). From Theorem 2.1, (2.1), (2.2) and (2.3) it is seen that we only have need to demonstrate

(1). $\bigcup_{1/2 \leq a < 1} N^a \not\subset D_\#$ and (2). $\bigcup_{1/2 \leq a < 1} N^a \subset UBC_0$. For (1), we take a function $f_5(z) = \sum_{k=0}^{\infty} \frac{z^{2^k}}{2^{k/2}}$. It is clear that f_5 is bounded and also in $N^a \setminus D_\#$, $a \in [1/2, 1)$, from [7, Theorem 1]. Thus (1) holds. For (2), we may fix $a \in [1/2, 1)$. For $w \in D$ and $f \in N^a$, we have

$$\begin{aligned} \int_D [f^\#(z)]^2 g(z, w) dm(z) &\leq \|f\|_{N^a}^2 \int_D \frac{g(z, w)}{(1 - |z|)^{2a}} dm(z) \\ &\leq 2^{2a} \|f\|_{N^a}^2 \int_D \frac{-(1 - |w|^2)^{2(1-a)} \log|z|}{(1 - |z|^2)^{2a} |1 - \bar{w}z|^{4(1-a)}} dm(z) \end{aligned}$$

By noting that $-\log|z| \leq 8(1 - |z|)$ for $|z| \geq \frac{1}{4}$, we also get

$$\int_{D \setminus D(0, \frac{1}{4})} \frac{-\log|z|}{(1 - |z|^2)^{2a} |1 - \bar{w}z|^{4(1-a)}} dm(z) \leq 8 \int_D \frac{(1 - |z|^2)^{1-2a}}{|1 - \bar{w}z|^{4(1-a)}} dm(z)$$

and

$$\int_{D(0, \frac{1}{4})} \frac{-\log|z|}{(1 - |z|^2)^{2a} |1 - \bar{w}z|^{4(1-a)}} dm(z) \leq \left(\frac{4}{3}\right)^{4-2a} 2\pi \int_0^{\frac{1}{4}} t \log \frac{1}{t} dt$$

Furthermore, we can obtain a constant $C > 0$ so that

$$\int_D [f^\#(z)]^2 g(z, w) dm(z) \leq C \|f\|_{N^a}^2 (1 - |w|^2)^{2(1-a)}. \quad (2.5)$$

Here we have used Lemma 4.2.2 in [10]. The above (2.5) gives $f \in UBC_0$, in other words, (2) holds.

(iii). We pick $f_5(z) = \sum_{k=1}^{\infty} \frac{z^{2k}}{2^k}$. By [7, Theorem 1] it follows that $f_5 \in \bigcap_{0 < a < 1} N_0^a$. However, it is very easy to observe that $f_5 \notin D_\#^\infty$. So, $\bigcap_{0 < a < 1} N_0^a \neq D_\#^\infty$.

3. CHARACTERIZATIONS OF N^a AND N_0^a .

In this section, we characterize functions in N^a and N_0^a for $a \in (0, 1]$ in terms of the weighted average, the pseudo-hyperbolic disk and the Green function, respectively. We use $|E|$ to denote the measure of the set $E \subseteq D$ relative to $dm(z)$, i.e., $|E| = \int_E dm(z)$.

THEOREM 3.1 Let $f \in M$, $a \in (0, 1]$ and $p \in (1, \infty)$. Then the following statements are equivalent:

- (i). $f \in N^a$.
- (ii). There is an $r_0 \in (0, 1)$ such that for any $r \in (0, r_0]$,

$$\sup_{w \in D} \frac{1}{|D(w, r)|^{1-a}} \int_{D(w, r)} [f^\#(z)]^2 dm(z) < \pi.$$

- (iii). There is an $r_0 \in (0, 1)$ such that for any $r \in (0, r_0]$,

$$\sup_{w \in D} \int_{D(w, r)} [f^\#(z)]^2 (1 - |z|^2)^{2(a-1)} dm(z) < \pi.$$

- (iv).

$$\sup_{w \in D} \int_D [f^\#(z)]^2 (1 - |z|^2)^{2(a-1)} g^p(z, w) dm(z) < \infty.$$

PROOF. We prove this theorem in accordance with the order (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

Step 1. (i) $\Rightarrow$ (ii). Let $f \in N^a$, then $\|f\|_{N^a} < \infty$. For $w \in D$ and $r \in (0, 1)$ we have

$$|D(w, r)| = \frac{\pi r^2 (1 - |w|^2)^2}{(1 - r^2 |w|^2)^2} \quad (3.1)$$

and

$$\int_{D(0, r)} \frac{1}{(1 - |z|^2)^{2a} |1 - \bar{w}z|^{2(2-a)}} dm(z) < \frac{\pi r}{(1 - r)^4}. \quad (3.2)$$

Further, we readily get

$$\begin{aligned} I_1(w, r) &= \frac{1}{|D(w, r)|^{1-\alpha}} \int_{D(w, r)} [f^\#(z)]^2 dm(z) \\ &\leq \frac{\pi(4r)^2}{(1-r)^4} \|f\|_{N^\alpha}^2. \end{aligned} \quad (3.3)$$

From (3.3) it follows that there exists an $r_0 \in (0, 1)$ for which $\sup_{w \in D} I_1(w, r) < \pi$ for any $r \in (0, r_0]$, i.e., (ii) holds.

Step 2. (ii) $\Rightarrow$ (iii). For $w \in D$ and $r \in (0, 1)$, by (3.1) we have

$$\begin{aligned} I_2(w, r) &= \int_{D(w, r)} [f^\#(z)]^2 (1 - |z|^2)^{2(\alpha-1)} dm(z) \\ &\leq \sup_{z \in D(w, r)} (1 - |z|^2)^{2(\alpha-1)} \int_{D(w, r)} [f^\#(z)]^2 dm(z) \\ &\leq \frac{\pi(4r)^{(1-\alpha)}}{(1-r)^{4(1-\alpha)}} \frac{1}{|D(w, r)|^{(1-\alpha)}} \int_{D(w, r)} [f^\#(z)]^2 dm(z). \end{aligned} \quad (3.4)$$

Once assuming (ii), we can choose $r_0 \in (0, 1)$ such that

$$\sup_{w \in D} \frac{1}{|D(w, r)|^{(1-\alpha)}} \int_{D(w, r)} [f^\#(z)]^2 dm(z) < \pi$$

for any $r \in (0, r_0]$. Further, when $r \in (0, r_0]$, we have

$$\sup_{w \in D} I_2(w, r) < \frac{\pi^2(4r)^{1-\alpha}}{(1-r)^{4(1-\alpha)}}. \quad (3.5)$$

Thus, there exists an $r_1 \in (0, r_0]$ such that $\sup_{w \in D} I_2(w, r) < \pi$ for any $r \in (0, r_1]$, and hence (iii) holds.

Step 3. (iii) $\Rightarrow$ (i). If (iii) holds, then there exists an $r_0 \in (0, 1)$ satisfying

$$C_0 = \sup_{w \in D} \frac{1}{\pi} \int_{D(w, r_0)} [f^\#(z)]^2 (1 - |z|^2)^{2(\alpha-1)} dm(z) < 1. \quad (3.6)$$

Consequently, for all $w \in D$,

$$S(r_0, f, w) = \frac{1}{\pi} \int_{D(w, r_0)} [f^\#(z)]^2 dm(z) \leq C_0 < 1. \quad (3.7)$$

Dufresnoy's lemma [5, Lemma II, p.216] then yields that

$$(1 - |w|^2)^{2\alpha} [f^\#(w)]^2 \leq \frac{S(r_0, f, w)(1 - |w|^2)^{2(\alpha-1)}}{r_0^2(1 - S(r_0, f, w))}. \quad (3.8)$$

Also,

$$\begin{aligned} &\int_{D(w, r_0)} [f^\#(z)]^2 (1 - |z|^2)^{2(\alpha-1)} dm(z) \\ &\geq \inf_{z \in D(w, r_0)} (1 - |z|^2)^{2(\alpha-1)} \int_{D(w, r_0)} [f^\#(z)]^2 dm(z) \\ &\geq (1 - r_0)^{4(1-\alpha)} (1 - |w|^2)^{2(\alpha-1)} \int_{D(w, r_0)} [f^\#(z)]^2 dm(z) \\ &= \pi(1 - r_0)^{4(1-\alpha)} (1 - |w|^2)^{2(\alpha-1)} S(r_0, f, w). \end{aligned} \quad (3.9)$$

From (3.7), (3.8) and (3.9) it derives that for all $w \in D$,

$$(1 - |w|^2)^a f^\#(w) \leq \left[\frac{C_0}{\pi r_0^2 (1 - C_0)(1 - r_0)^{4(1-a)}} \right]^{\frac{1}{2}}, \quad (3.10)$$

namely, $f \in N^a$.

Step 4. (i) $\Rightarrow$ (iv). Under $f \in N^a$, we have

$$\begin{aligned} I_3(w) &= \int_D [f^\#(z)]^2 (1 - |z|^2)^{2(a-1)} g^p(z, w) dm(z) \\ &\leq C_1 \|f\|_{N^a}^2, \end{aligned} \quad (3.11)$$

where $C_1 = 2\pi \int_0^1 t(1 - t^2)^{-2} \log^p \frac{1}{t} dt < \infty$ for $p \in (1, \infty)$. So, (iv) follows.

Step 5. (iv) $\Rightarrow$ (iii). If (iv) holds, then, for $w \in D$ and $r \in (0, 1)$, we have

$$\int_{D(w, r)} [f^\#(z)]^2 (1 - |z|^2)^{2(a-1)} dm(z) \leq -\frac{I_3(w)}{\log r}. \quad (3.12)$$

That is to say, we can choose an $r_0 \in (0, 1)$ so that $\sup_{w \in D} I_2(w, r) < \pi$ for any $r \in (0, r_0]$.

This completes the proof.

For N_0^a we have a similar result.

THEOREM 3.2 Let $f \in M$, $a \in (0, 1]$ and $p \in (1, \infty)$. Then the following statements are equivalent:

- (i). $f \in N_0^a$.
- (ii). There is an $r_0 \in (0, 1)$ such that for any $r \in (0, r_0]$,

$$\lim_{|w| \rightarrow 1} \frac{1}{|D(w, r)|^{1-a}} \int_{D(w, r)} [f^\#(z)]^2 dm(z) = 0.$$

- (iii). There is an $r_0 \in (0, 1)$ such that for any $r \in (0, r_0]$,

$$\lim_{|w| \rightarrow 1} \int_{D(w, r)} [f^\#(z)]^2 (1 - |z|^2)^{2(a-1)} dm(z) = 0.$$

- (iv).

$$\lim_{|w| \rightarrow 1} \int_D [f^\#(z)]^2 (1 - |z|^2)^{2(a-1)} g^p(z, w) dm(z) = 0.$$

PROOF. We show this theorem according to the order (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

Step 1. (i) $\Rightarrow$ (ii). Suppose that $f \in N_0^a$. Then for any compact subset $E \subset D$ and all $z \in E$, such a function f satisfies

$$\lim_{|w| \rightarrow 1} (1 - |\phi_w(z)|^2)^a f^\#(\phi_w(z)) = 0. \quad (3.13)$$

Thus, for any $\epsilon > 0$, there exists a $\rho \in (0, 1)$ such that for $|w| > \rho$,

$$\int_{D(w, r)} [f^\#(z)]^2 dm(z) \leq \epsilon \int_{D(0, r)} \frac{|1 - \bar{w}z|^{4(a-1)}}{(1 - |w|^2)^{2(a-1)}(1 - |z|^2)^{2a}} dm(z).$$

As its consequence,

$$I_1(w, r) \leq \frac{\epsilon(\pi r)^a}{(1 - r)^{1-4a}}, \quad (3.14)$$

and hence it turns out that there is an $r_0 \in (0, 1)$ to make $I_1(w, r) < \epsilon$ for all $w \in D \setminus D(0, \rho)$ and any $r \in (0, r_0]$. i.e., (ii) holds.

Step 2. (ii) $\Rightarrow$ (iii). This follows readily from (3.4).

Step 3. (iii) $\Rightarrow$ (i). Assuming that $f \in M$ is of (iii), for any $\epsilon \in (0, 1)$, we can find $\rho \in (0, 1)$ to make $I_2(w, r_0) < \pi\epsilon$, and consequently $S(r_0, f, w) \leq \epsilon < 1$ for all $w \in D \setminus D(0, \rho)$. Combining (3.8) and (3.9) we get

$$(1 - |w|^2)^a f^\#(w) \leq \left[\frac{\epsilon}{(1 - r_0)^{(5-4a)}} \right]^{\frac{1}{2}} \quad (3.15)$$

for all $w \in D \setminus D(0, \rho)$. Therefore $f \in N_0^a$.

Step 4. (i) $\Rightarrow$ (iv). Provided that (i) is true. Since $C_1 < \infty$, for any $\epsilon > 0$ there is an $r_2 \in (0, 1)$ such that

$$\int_{D \setminus D(0, r_2)} \frac{\log^p \frac{1}{|z|}}{(1 - |z|^2)^2} dm(z) < \epsilon. \quad (3.16)$$

Also, for this r_2 and all $w \in D$,

$$\begin{aligned} I_3(w) &= \left(\int_{D(0, r_2)} + \int_{D \setminus D(0, r_2)} \right) [f^\#(\phi_w(z))]^2 \frac{(1 - |w|^2)^{2a} (1 - |z|^2)^{2(a-1)}}{|1 - \bar{w}z|^{4a}} \log^p \frac{1}{|z|} dm(z) \\ &= \left(\int_{D(0, r_2)} + \int_{D \setminus D(0, r_2)} \right) (\dots) dm(z). \end{aligned}$$

From the condition: $f \in N_0^a$ it follows that there exists a $\rho_1 \in (0, 1)$ such that for $|w| > \rho_1$,

$$\int_{D(0, r_2)} (\dots) dm(z) \leq \left(\frac{16}{15} \right)^2 \epsilon^2 \int_0^{r_2} \log^p \frac{1}{t} dt \quad (3.17)$$

and

$$\int_{D \setminus D(0, r_2)} (\dots) dm(z) \leq \|f\|_{N^a}^2 \int_{D \setminus D(0, r_2)} \frac{\log^p \frac{1}{|z|}}{(1 - |z|^2)^2} dm(z). \quad (3.18)$$

Combining (3.16), (3.17) and (3.18) deduces (iii) right away.

Step 5. (iv) $\Rightarrow$ (iii). This is a simple consequence of (3.12).

This completes the proof.

REMARK. (i). A special case of (i) $\Leftrightarrow$ (iii) in Theorem 3.1 was stated by Wulan and Yan [6]. (ii). The case: $a = 1$ of (i) $\Leftrightarrow$ (iv) in Theorem 3.1 and in Theorem 3.2 was given by Aulaskari and Lappan [3]. (iii). The ideas and examples of this paper are suitable for the α -Bloch and little α -Bloch functions (see [7, 9]). (iv). It is an open question as to which of the results from this paper are valid for $a \in (1, \infty)$. Similar questions may also be asked about corresponding classes of harmonic functions (Cf [3]).

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