FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS WITH APPLICATIONS TO THE SOLUTIONS OF FUNCTIONAL EQUATIONS ARISING IN DYNAMIC PROGRAMMINGS

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ABSTRACT. Some common fixed point theorems for compatible mappings are shown. As an application, the existence and uniqueness of common solutions for a class of functional equations arising in dynamic programmings are discussed.

KEY WORDS AND PHRASES: Common fixed point, compatible mapping, dynamic programming

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1. INTRODUCTION

In [1] the concept of compatible mappings was introduced as a generalization of commuting mappings and further investigation was given in [2-9].

The purpose of this paper is to prove some common fixed point theorems for compatible mappings, which generalized some recent results of [4, 10-13]. As an application, we use the results presented to study the existence and uniqueness problem of a common solution for a class of functional equations arising in dynamic programmings, which generalized the corresponding results of [14, 15].

2. FIXED POINT THEOREMS

DEFINITION 2.1. Self mappings A and S of a metric space (X, d) are called compatible, if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X .

It is clear that commuting mappings and weakly commuting mappings are all compatible mappings, but the converse is false (see [1, 4]).

LEMMA 2.2 [1, 4] If A and S are compatible self mappings of a metric space (X, d) and $\lim_n Sx_n = \lim_n Ax_n = t$ for some t in X , then $\lim_n ASx_n = St$ if S is continuous.

The following theorem can be obtained from Theorem 8 in [16].

THEOREM 2.3. Let (X, d) be a complete metric space and A, B, S and T are self mappings of X . Suppose that S and T are continuous, $A(X) \subset T(X)$, $B(X) \subset S(X)$, and that A, S and B, T are compatible and satisfy the following condition:

$$d(Ax, By) \leq \Phi(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \\ \frac{1}{2} [d(Sx, By) + d(Ty, Ax)]\}), \quad \forall x, y \in X, \quad (2.1)$$

where $\Phi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, upper semicontinuous and $\Phi(t) < t$ for all $t > 0$.

Then A , B , S and T have a unique common fixed point in X .

We merely state the proof for convenience.

PROOF. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, we can choose a sequence $\{x_n\}$ in X such that $Sx_{2n} = Bx_{2n-1}$ and $Tx_{2n-1} = Ax_{2n-2}$ for all n in the set $\mathbb{N}$ of all positive integers. Let

$$\left. \begin{aligned} y_{2n-1} &= Tx_{2n-1} = Ax_{2n-2} \\ y_{2n} &= Sx_{2n} = Bx_{2n-1} \end{aligned} \right\} (n \in \mathbb{N}). \quad (2.2)$$

As in [10], we can prove that $\{y_n\}$ is a Cauchy sequence in X . Letting $y_n \rightarrow y_* \in X$ ($n \rightarrow \infty$), we know that $\{y_{2n}\}$ and $\{y_{2n-1}\}$ converge to y_* too.

Since A and S , B and T are both compatible, it follows from the continuity of S and T , (2.2) and Lemma 2.2 that

$$Ty_{2n-1} \rightarrow Ty_*, \quad By_{2n-1} \rightarrow Ty_*, \quad Sy_{2n} \rightarrow Sy_*, \quad Ay_{2n} \rightarrow Sy_*. \quad (2.3)$$

By (2.1) and (2.2) we have

$$\begin{aligned} d(Ay_{2n}, By_{2n-1}) &\leq \Phi(\max\{d(Sy_{2n}, Ty_{2n-1}), d(Sy_{2n}, Ay_{2n}), \\ &\quad d(Ty_{2n-1}, By_{2n-1}), \frac{1}{2}[d(Sy_{2n}, By_{2n-1}) + d(Ty_{2n-1}, Ay_{2n})]\}). \end{aligned}$$

By the upper semicontinuity of $\Phi(t)$ and (2.3) we have

$$\begin{aligned} d(Sy_*, Ty_*) &\leq \Phi(\max\{d(Sy_*, Ty_*), 0, 0, d(Sy_*, Ty_*)\}) \\ &= \Phi(d(Sy_*, Ty_*)). \end{aligned}$$

This implies that $Sy_* = Ty_*$.

Similarly, from (2.1), (2.2) and (2.3) we can obtain

$$Sy_* = By_*, \quad Ty_* = Ay_*.$$

Hence we have

$$Ay_* = By_* = Sy_* = Ty_*. \quad (2.4)$$

From (2.1) and (2.2) we have

$$\begin{aligned} d(Ax_{2n}, By_*) &\leq \Phi(\max\{d(Sx_{2n}, Ty_*), d(Sx_{2n}, Ax_{2n}), \\ &\quad d(Ty_*, By_*), \frac{1}{2}[d(Sx_{2n}, By_*) + d(Ty_*, Ax_{2n})]\}), \end{aligned}$$

and then

$$d(y_*, By_*) \leq \Phi(d(y_*, By_*)).$$

Hence we have $y_* = By_* = Ay_* = Sy_* = Ty_*$.

The uniqueness is obvious. This completes the proof.

DEFINITION 2.4. A metric space (X, d) is (metrical) convex, if for each $x, y \in X$ with $x \neq y$, there exists a $z \in X$, $x \neq z \neq y$, such that

$$d(x, z) + d(z, y) = d(x, y).$$

LEMMA 2.5 [17]. Let K be a closed subset of a complete convex metric space X . If $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

DEFINITION 2.6. Let (X, d) be a metric space, $K \subset X$ and $A, S : K \rightarrow X$. The pair of mappings A and S is called compatible, if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in K such that $Ax_n, Sx_n \in K$ and $\lim_n Ax_n = \lim_n Sx_n = t \in K$.

LEMMA 2.7. Let (X, d) be a metric space, $K \subset X$ and $A, S: K \rightarrow X$. If A and S are compatible mappings, $Ax_n, Sx_n \in K$ and $\lim_n Ax_n = \lim_n Sx_n = t$ for some $t \in K$, then $\lim_n ASx_n = St$ if S is continuous.

PROOF. It is obvious from Definition 2.6

THEOREM 2.8. Let (X, d) be a complete convex metric space and K a nonempty closed subset of X . Suppose that S and T are continuous mappings from X into X with $\partial K \subset S(K) \cap T(K)$ and that $A, B: K \rightarrow X$ are continuous mappings with $A(K) \cap K \subset S(K)$, $B(K) \cap K \subset T(K)$. Suppose further that the pairs of mappings A, T and B, S are compatible and satisfying

$$d(Ax, By) \leq \Phi(d(Tx, Sy)), \forall x, y \in K, \quad (2.5)$$

where $\Phi: [0, \infty) \rightarrow [0, \infty)$ is nondecreasing upper semi-continuous and $\sum \Phi^n(t) < \infty$ for all $t \geq 0$

If for $x \in K$, $Tx \in \partial K$ implies $Ax, Bx \in K$ and $Sx \in \partial K$ implies $Ax, Bx \in K$, then there exists a $z \in K$ such that

$$z = Tz = Sz = Az = Bz.$$

If $Tv = Sv = Av = Bv$, then $Tz = Tv$

PROOF. Let $p \in \partial K$. Using Lemma 2.5 and the proof of [11] we can choose two sequences $\{p_n\}_{n \in \mathbb{N}}$ and $\{p'_n\}_{n \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$, $p_n \in K$, $p'_{2n-1} = Ap_{2n}$, $p'_{2n} = Bp_{2n-1}$ and the following implications hold:

- (i) If $p'_{2n} \in K$, then $p'_{2n} = Tp_{2n}$, if $p'_{2n} \notin K$, then $Tp_{2n} \in \partial K$ and
- $$d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, Bp_{2n-1}) = d(Sp_{2n-1}, Bp_{2n-1})$$
- (ii) If $p'_{2n+1} \in K$, then $p'_{2n+1} = Sp_{2n+1}$, if $p'_{2n+1} \notin K$, then $Sp_{2n+1} \in \partial K$ and
- $$d(Tp_{2n}, Sp_{2n+1}) + d(Sp_{2n+1}, Ap_{2n}) = d(Tp_{2n}, Ap_{2n})$$

Further, as in [3] we can prove that

$$\left. \begin{aligned} d(Tp_{2n}, Sp_{2n+1}) &\leq \Phi^{n-1}(r) \\ d(Sp_{2n+1}, Tp_{2n+2}) &\leq \Phi^n(r) \end{aligned} \right\} \quad (n \in \mathbb{N}), \quad (2.6)$$

where $r = \max\{d(Tp_2, Sp_3), d(Tp_2, Sp_1)\}$.

This implies that for any $n \in \mathbb{N}$,

$$d(Tp_{2n}, Tp_{2n+2}) \leq \Phi^{n-1}(r) + \Phi^n(r).$$

Hence the sequence $\{Tp_{2n}\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Since X is complete and K is closed, it follows that there exists a $z \in K$ such that $z = \lim_n Tp_{2n}$. From (2.6) we have

$$z = \lim_n Tp_{2n} = \lim_n Sp_{2n+1}.$$

Now we prove that $z = Tz = Sz = Az = Bz$. It is obvious that there exists a sequence $\{n_k\} \subset \mathbb{N}$ such that $Tp_{2n_k} = Bp_{2n_k-1}$, or $Sp_{2n_k-1} = Ap_{2n_k-2}$, $\forall k \in \mathbb{N}$. Without loss of generality, we can suppose that $Tp_{2n_k} = Bp_{2n_k-1}$, $\forall k \in \mathbb{N}$. From (2.5) we have

$$\begin{aligned} d(STp_{2n_k}, Az) &\leq d(SBp_{2n_k-1}, BS p_{2n_k-1}) + d(BS p_{2n_k-1}, Az) \\ &\leq d(SBp_{2n_k-1}, BS p_{2n_k-1}) + \Phi(d(SS p_{2n_k-1}, Tz)). \end{aligned}$$

Since B, S are compatible and S is continuous, we have

$$d(Sz, Az) \leq \Phi(d(Sz, Tz)). \quad (2.7)$$

From (2.5) we have

$$\begin{aligned} d(Ap_{2n_k}, Tp_{2n_k}) &= d(Ap_{2n_k}, Bp_{2n_k-1}) \\ &\leq \Phi(d(Sp_{2n_k-1}, Tp_{2n_k})). \end{aligned}$$

By the upper semi-continuity of $\Phi(t)$, it follows that

$$\lim_k Ap_{2n_k} = z. \quad (2.8)$$

Using (2.5) we have

$$d(Ap_{2n_k}, BSp_{2n_k-1}) \leq \Phi(d(Tp_{2n_k}, SS_{p_{2n_k-1}})).$$

Since B, S are compatible and S is continuous, it follows from (2.8) and Lemma 2.7 that

$$d(z, Sz) \leq \Phi(d(z, Sz)).$$

This implies that $d(z, Sz) = 0$, i.e. $z = Sz$.

Since A, T are compatible and A and T are continuous, from (2.8) and Lemma 2.7 we have

$$Az = \lim_k ATp_{2n_k} = Tz.$$

In view of (2.7) we have

$$d(Sz, Tz) \leq \Phi(d(Sz, Tz))$$

and so

$$z = Sz = Tz = Az.$$

Besides, from (2.5) we have

$$d(Az, Bz) \leq \Phi(d(Sz, Tz)) = \Phi(0) = 0.$$

Hence

$$z = Tz = Sz = Az = Bz.$$

Finally, if $Tv = Sv = Av = Bv$, then

$$d(Tv, Sz) = d(Av, Bz) \leq \Phi(d(Sz, Tv))$$

and so $Tv = Sz = Tz$.

This completes the proof of Theorem 2.8.

As an immediate consequence we can obtain the following result.

THEOREM 2.9. Let (X, d) be a complete convex metric space, K a nonempty closed subset of X , and S and T continuous mappings from X into X such that $\partial K \subset S(K) \cap T(K)$. Suppose that for every $n \in \mathbb{N}$, $A_n : K \rightarrow X$ is a continuous mapping with $A_{2n}(K) \cap K \subset T(K)$ and $A_{2n-1}(K) \cap K \subset S(K)$, and that the pairs of mappings A_{2n-1}, T and A_{2n}, S are compatible such that for any $n \in \mathbb{N}$

$$d(A_n x, A_{n+1} y) \leq \Phi(d(Tx, Sy)), \forall x, y \in K,$$

where $\Phi(t)$ is the same as in Theorem 2.8.

If for every $n \in \mathbb{N}$ and $x \in K$,

$$Tx \in \partial K \text{ implies } A_n x \in K \text{ and } Sx \in \partial K \text{ implies } A_n x \in K,$$

then there exists a $z \in K$ such that

$$z = Tz = Sz = A_n z, \quad \forall n \in \mathbb{N},$$

and if $Tv = Sv = A_n v$ for every $n \in \mathbb{N}$, then $Tz = Tv$.

REMARK 2.10. Theorem 2.9 is a generalization of Theorem 1 in [11]

3. APPLICATIONS

Throughout this section we assume that X and Y are Banach spaces, $S \subset X$ is a state space, $D \subset Y$ a decision space and $\mathbb{R} = (-\infty, +\infty)$. We denote by $B(S)$ the set of all bounded real-valued functions defined on S .

As suggested in Bellman and Lee [18], the basic form of the functional equations of dynamic programming is

$$f(x) = \text{opt}_y H(x, y, f(T(x, y))),$$

where x and y represent the state and decision vectors respectively, T represents the transformation of the process, and $f(x)$ represents the optimal return function with initial state x (here opt denotes $\max$ or $\min$)

In this section, we shall study the existence and uniqueness of a common solution of the following functional equations arising in dynamic programmings

$$f(x) = \sup_{y \in D} H_1(x, y, f(T(x, y))), \quad x \in S, \quad (3.1)$$

$$g(x) = \sup_{y \in D} H_2(x, y, g(T(x, y))), \quad x \in S, \quad (3.2)$$

$$p(x) = \sup_{y \in D} F_1(x, y, p(T(x, y))), \quad x \in S, \quad (3.3)$$

$$q(x) = \sup_{y \in D} F_2(x, y, q(T(x, y))), \quad x \in S, \quad (3.4)$$

where $T : S \times D \rightarrow S$, H_i and $F_i : S \times D \times \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$.

THEOREM 3.1. Suppose that the following conditions are satisfied:

(i) H_i and F_i are bounded, $i = 1, 2$.

(ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq \Phi(\max\{|T_1 h(t) - T_2 k(t)|, |T_1 h(t) - A_1 h(t)|, |T_2 k(t) - A_2 k(t)|, \frac{1}{2}(|T_1 h(t) - A_2 k(t)| + |T_2 k(t) - A_1 h(t)|)\})$,

for all $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$, where Φ is the same as in Theorem 2.3, and the mappings A_i and T_i are defined as follows:

$$A_i h(x) = \sup_{y \in D} H_i(x, y, h(T(x, y))), \quad x \in S, \quad h \in B(S) \quad \text{and}$$

$$T_i k(x) = \sup_{y \in D} F_i(x, y, k(T(x, y))), \quad x \in S, \quad k \in B(S), \quad i = 1, 2.$$

(iii) For any $\{k_n\} \subset B(S)$ and $k \in B(S)$,

$$\limsup_n \sup_{x \in S} |k_n(x) - k(x)| = 0 \quad \text{implies} \quad \limsup_n \sup_{x \in S} |T_i k_n(x) - T_i k(x)| = 0, \quad i = 1, 2.$$

(iv) For any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$ such that

$$A_1 h(x) = T_2 k_1(x), \quad A_2 h(x) = T_1 k_2(x), \quad x \in S.$$

(v) For any $\{k_n\} \subset B(S)$, if there exists $h \in B(S)$ such that

$$\limsup_n \sup_{x \in S} |A_i k_n(x) - h(x)| = \limsup_n \sup_{x \in S} |T_i k_n(x) - h(x)| = 0,$$

then

$$\limsup_n \sup_{x \in S} |A_i T_i k_n(x) - T_i A_i k_n(x)| = 0, \quad i = 1, 2.$$

Then the system of functional equations (3.1)-(3.4) has a unique common solution in $B(S)$.

PROOF. For any $h, k \in B(S)$, let

$$d(h, k) = \sup \{|h(x) - k(x)| : x \in S\},$$

then $(B(S), d)$ is a complete metric space. From (i)-(v) we know that A_i and T_i are self mappings of $B(S)$, T_i are continuous, $A_1(B(S)) \subset T_2(B(S))$, $A_2(B(S)) \subset T_1(B(S))$, and the pair of mappings A_i, T_i are compatible, $i = 1, 2$.

Let h_1, h_2 be any two points of $B(S)$, let $x \in S$ and η be any positive number, there exist y_1 , and y_2 in D such that

$$\left. \begin{aligned} A_i h_i(x) &< H_i(x, y_i, h_i(x_i)) + \eta \\ \text{where } x_i &= T(x, y_i) \end{aligned} \right\} \quad (i = 1, 2). \quad (3.5)$$

Also we have

$$A_1 h_1(x) \geq H_1(x, y_2, h_1(x_2)), \quad (3.6)$$

$$A_2 h_2(x) \geq H_2(x, y_1, h_2(x_1)). \quad (3.7)$$

From (3.5), (3.7) and (ii) we have

$$\begin{aligned} A_1 h_1(x) - A_2 h_2(x) &< H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \eta \\ &\leq |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \eta \\ &\leq \Phi(\max\{|T_1 h_1(x_1) - T_2 h_2(x_1)|, |T_1 h_1(x_1) - A_1 h_1(x_1)|, \\ &\quad |T_2 h_2(x_1) - A_2 h_2(x_1)|, \frac{1}{2} [|T_1 h_1(x_1) - A_2 h_2(x_1)| \\ &\quad + |T_2 h_2(x_1) - A_1 h_1(x_1)|]\}) + \eta \\ &\leq \Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), \\ &\quad d(T_2 h_2, A_2 h_2), \frac{1}{2} [d(T_1 h_1, A_2 h_2) \\ &\quad + d(T_2 h_2, A_1 h_1)]\}) + \eta. \end{aligned}$$

Similarly from (3.5), (3.6) and (ii) we have

$$\begin{aligned} A_1 h_1(x) - A_2 h_2(x) &\geq -\Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), \\ &\quad d(T_2 h_2, A_2 h_2), \frac{1}{2} [d(T_1 h_1, A_2 h_2) \\ &\quad + d(T_2 h_2, A_1 h_1)]\}) - \eta. \end{aligned}$$

Hence we have

$$\begin{aligned} |A_1 h_1(x) - A_2 h_2(x)| &\leq \Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), \\ &\quad d(T_2 h_2, A_2 h_2), \frac{1}{2} [d(T_1 h_1, A_2 h_2) \\ &\quad + d(T_2 h_2, A_1 h_1)]\}) + \eta. \end{aligned} \quad (3.8)$$

Since (3.8) is true for any $x \in S$ and η is any positive number, we have

$$\begin{aligned} d(A_1 h_1, A_2 h_2) &\leq \Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), \\ &\quad d(T_2 h_2, A_2 h_2), \frac{1}{2} [d(T_1 h_1, A_2 h_2), \\ &\quad + d(T_2 h_2, A_1 h_1)]\}). \end{aligned}$$

Therefore by Theorem 2.3, A_1, A_2, T_1 and T_2 have a unique common fixed point $h^* \in B(S)$, i.e. $h^*(x)$ is a unique common solution of functional equations (3.1) - (3.4). This completes the proof.

The following result is an immediate consequence of Theorem 2.3 and Theorem 3.1

THEOREM 3.2. Suppose that the following conditions are satisfied:

- (i) H_i is bounded, $i = 1, 2$;
- (ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq \Phi(\max\{|h(t) - k(t)|, |h(t) - A_1 h(t)|, |k(t) - A_2 k(t)|, \frac{1}{2} [|h(t) - A_2 k(t)| + |k(t) - A_1 h(t)|]\})$

for all $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$, where Φ is the same as in Theorem 2.3 and A_i is defined by

$$A_i h(x) = \sup_{y \in D} H_i(x, y, h(T(x, y))), \quad x \in S, \quad h \in B(S), \quad i = 1, 2.$$

Then the functional equations (3.1) and (3.2) have a unique common solution in $B(S)$

REMARK 3.3. Theorem 3.2 is a generalization of Theorem 2.1 in [15].

THEOREM 3.4. Suppose that the following conditions are satisfied.

- (i) H_i and F_i are bounded, $i = 1, 2$,
- (ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq \Phi(|T_1 h(t) - T_2 k(t)|)$ for all $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$, where Φ is the same as in Theorem 2.8 and T_i is defined as in Theorem 3.1, $i = 1, 2$;
- (iii) For any $\{k_n\} \subset B(S)$ and $k \in B(S)$,

$$\limsup_n \sup_{x \in S} |k_n(x) - k(x)| = 0 \quad \text{implies} \quad \limsup_n \sup_{x \in S} |T_i k_n(x) - T_i k(x)| = 0$$

and

$$\limsup_n \sup_{x \in S} |A_i k_n(x) - A_i k(x)| = 0, \quad i = 1, 2,$$

where A_i is defined as in Theorem 3.1, $i = 1, 2$;

- (iv) For any $h \in B(S)$ such that $\sup_{x \in S} |h(x)| = 1$, there exist $k_1, k_2 \in B(S)$ such that

$$\sup_{x \in S} |k_i(x)| \leq 1 \quad \text{and} \quad T_i k_i(x) = h(x), \quad x \in S, \quad i = 1, 2;$$

- (v) For any $h \in B(S)$ such that $\sup_{x \in S} |h(x)| \leq 1$, there exist $k_1, k_2 \in B(S)$ such that

$$\sup_{x \in S} |k_i(x)| \leq 1, \quad i = 1, 2, \quad A_1 h(x) = T_2 k_1(x) \quad \text{and} \quad A_2 h(x) = T_1 k_2(x), \quad x \in S;$$

- (vi) For any $h \in B(S)$ such that $\sup_{x \in S} |h(x)| \leq 1$,

$$\sup_{x \in S} |T_i h(x)| = 1 \quad \text{implies} \quad \sup_{x \in S} |A_j h(x)| \leq 1, \quad i, j = 1, 2;$$

- (vii) For any $\{k_n\} \subset B(S)$, if there exists $h \in B(S)$ such that $\sup_{x \in S} |T_i k_n(x)| \leq 1$ and

$$\limsup_n \sup_{x \in S} |A_i k_n(x) - h(x)| = \limsup_n \sup_{x \in S} |T_i k_n(x) - h(x)| = 0,$$

then

$$\limsup_n \sup_{x \in S} |A_i T_i k_n(x) - T_i A_i k_n(x)| = 0, \quad i = 1, 2.$$

Then the system of functional equations (3.1) - (3.4) have a unique common solution $h^* \in B(S)$ and $\sup_{x \in S} |h^*(x)| \leq 1$.

PROOF. Let us consider $B(S)$ as a Banach space of all bounded real-valued functions defined on S with a supremum norm, and K the closed unit ball in $B(S)$. By conditions (i)-(vii) we know that $A_i : K \rightarrow B(S)$ and $T_i : B(S) \rightarrow B(S)$, $i = 1, 2$, satisfy all of the conditions of Theorem 2.8 and have a unique common fixed point $h^* \in K$, i.e., $h^*(x)$ is a unique common solution of functional equations (3.1) - (3.4).

REMARK 3.5. Theorem 3.4 is a generalization of Theorem 3.2 in [14].

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