TWO ELEMENTARY COMMUTATIVITY THEOREMS FOR GENERALIZED BOOLEAN RINGS

VISHNU GUPTA

Department of Mathematics
M.D. University, P.G. Regional Centre
Rewari (Haryana), India

(Received September 9, 1991 and in revised form April 17, 1992)

ABSTRACT. In this paper we prove that if R is a ring with 1 as an identity element in which $x^m - x^n \in Z(R)$ for all $x \in R$ and fixed relatively prime positive integers m and n , one of which is even, then R is commutative. Also we prove that if R is a 2-torsion free ring with 1 in which $(x^{2^k})^{n+1} - (x^{2^k})^n \in Z(R)$ for all $x \in R$ and fixed positive integer n and non-negative integer k , then R is commutative.

KEY WORDS AND PHRASES. Commutator, 2-torsion free ring.

1991 AMS SUBJECT CLASSIFICATION CODE. 16A70.

1. INTRODUCTION.

Throughout this paper, R is an associative ring with 1 as an identity element. We denote the centre of R by $Z(R)$ and the commutator $xy - yx$ by $[x, y]$. Recently, Quadri and Ashraf [1] proved that if R is a ring in which $x^{n+1} - x^n \in Z(R)$ for all $x \in R$ and fixed positive integer n , then R is commutative. In this paper, we generalize this result.

2. MAIN RESULTS.

We start with the following lemma of Bell [2].

LEMMA 2.1. Let $w \in R$. If for each $x \in R$ there exist relatively prime positive integers $n = n(x)$ and $m = m(x)$ such that

$$[w, x^n] = [w, x^m] = 0, \text{ then } w \in Z(R).$$

THEOREM 2.1. If R is a ring with $x^m - x^n \in Z(R)$ for all $x \in R$ and fixed relatively prime positive integers m and n , one of which is even, then R is commutative.

PROOF. Let

$$x^m - x^n \in Z(R) \quad \text{for all } x \in R. \quad (2.1)$$

Assume m is even and n is odd. Using both x and $-x$ in (2.1) and then adding and subtracting, we get $2x^m \in Z(R)$ and $2x^n \in Z(R)$. Thus $[x^m, 2y] = [x^n, 2y] = 0$ for all $x, y \in R$; and by Lemma 2.1 $2y \in Z(R)$ for all $y \in R$. Now we replace x by $x+1$ to obtain

$$\left[\sum_{r=1}^{m-1} \binom{m}{r} x^{m-r}, y \right] = \left[\sum_{r=1}^{n-1} \binom{n}{r} x^{n-r}, y \right]$$

and since m is even and n is odd and $[2x, y] = 0$, we get $[x^{2p(x)} - x, y] = 0$ for some $p(x) \in Z[x]$. Now

the theorem follows from Herstein's result [3].

In Theorem 2.1, all the hypotheses are essential. If both m and n are odd or if one of m and n is even and the other odd, but they are not relatively prime; or if both m and n are even; or if R is a ring without the identity element 1 in the hypotheses of the theorem, then R need not be commutative.

EXAMPLE 2.1. It can be shown easily that

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in GF(3) \right\}$$

is a ring with identity element, in which

(i) $x^3 - x^9 \in Z(R)$

(ii) $x^3 - x^6 \in Z(R)$

(iii) $x^4 - x^{10} \in Z(R)$

for all $x \in R$, but R is not commutative.

EXAMPLE 2.2.

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in GF(2) \right\}$$

is a ring with identity element in which

(i) $x^5 - x^9 \in Z(R)$

(ii) $x^4 - x^8 \in Z(R)$

for all $x \in R$, but R is not commutative.

EXAMPLE 2.3.

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in GF(3) \right\}$$

is a ring without identity element with $x^3 - x^4 \in Z(R)$ for all $x \in R$, but R is not commutative.

We state the following lemma which can be proved easily.

LEMMA 2.2. If $t = 2^k n$ where k and n are positive integers, then

$$\left\{ \binom{t+2^k}{(2r-1)+2^k} - \binom{t}{2r-1} \right\} \text{ and } \binom{t+2^k}{2r-1} \text{ are multiples of } 2^k \quad \text{for } r = 1, 2, 3, \dots, \frac{t}{2}.$$

Now we give the following theorem which generalizes the theorem of Quadri and Ashraf [1] for 2-torsion free rings.

THEOREM 2.2. If R is a 2-torsion free ring with $(x^{2^k})^{n+1} - (x^{2^k})^n \in Z(R)$ for all $x \in R$ and fixed non-negative integer k and positive integer n , then R is commutative.

PROOF. If $k = 0$ then result follows from Theorem 2.1. Let $k > 0$ and

$$[x^{t+2^k}, y] = [x^t, y] \text{ for all } x, y \in R \text{ where } t = 2^k n$$

Now we replace x by $x+1$ to obtain

$$\left[\sum_{r=1}^{t+2^k-1} \binom{t+2^k}{r} x^{t+2^k-r}, y \right] = \left[\sum_{r=1}^{t-1} \binom{t}{r} x^{t-r}, y \right] \quad (2.2)$$

Next we replace x by $-x$ in (2.2) and subtract the result from (2.2) and use the fact that R is 2-torsion free to get

$$\left[\sum_{r=1}^{2^k-1} \binom{t+2^k}{2r-1} x^{t+2^k-(2r-1)} + \sum_{r=1}^{t/2} \left\{ \binom{t+2^k}{(2r-1)+2^k} - \binom{t}{2r-1} \right\} x^{t-(2r-1)}, y \right] = 0$$

By Lemma 2.2 and the fact that R is 2-torsion free, we get $[x^2 p(x) - x, y] = 0$ for all $x, y \in R$ and some $p(x) \in Z[x]$. Now R is commutative.

All the hypotheses of Theorem 2.2 are essential. In Example 2.1, R is a 2-torsion free ring with identity element in which $(x^{2^k})^m - (x^{2^k})^n \in Z(R)$ ($k=2, m=4, n=7$) for all $x, y \in R$ and m and n are relatively prime positive integers and one of them is even, but R is not commutative. In Example 2.2, R is a 2-torsion ring with identity element in which $(x^{2^k})^{n+1} - (x^{2^k})^n \in Z(R)$ ($k=2, n=1$) for all $x \in R$, but R is not commutative. In Example 2.3, R is a 2-torsion free ring without identity element in which $(x^{2^k})^{n+1} - (x^{2^k})^n \in Z(R)$ ($k=2$ and $n=1$) for all $x \in R$, but R is not commutative.

ACKNOWLEDGEMENT. The author expresses his sincere thanks to the referee for many helpful suggestions to modify the proof of Theorem 2.1.

REFERENCES

1. QUADRI, M.A. & ASHRAF, MOHD., Commutativity of generalized Boolean rings, Publ. Math. (Debrecen) 35 (1988), 73-75.
2. BELL, H.E., On rings with commuting powers, Math. Japonica 24 (1979), 473-478.
3. HERSTEIN, I.N., Two remarks on the commutativity of rings, Cand. J. Math. 7 (1955), 411-412.
4. HERSTEIN, I.N., A generalization of theorem of Jacobson, Amer. J. Math. 73 (1951), 755-762.