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ON X-VALUED SEQUENCE SPACES

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ABSTRACT. Certain spaces of X-valued sequences are introduced and some of their properties are investigated. Köthe- Toeplitz duals of these spaces are examined.

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1. INTRODUCTION AND BACKGROUND.

Let c_0, c, l_∞ and s respectively denote the spaces of null sequences, convergent sequences, bounded sequences and all sequences. Let X be a complex linear space with zero element θ and $X = (X, \|.\|)$ be a seminormed space. We may define $c_0(X)$ the null X-valued sequences, $c(X)$ the convergent X-valued sequences, $l_\infty(X)$ the bounded X-valued sequences and $s(X)$ the vector space of all X-valued sequences. If we take $X = C$ the set of complex numbers these spaces reduce to the already familiar spaces c_0, c, l_∞ and s respectively. These spaces of X-valued sequences have been studied by Maddox[2,3], Rath[5], Pehlivan[4] and others. We take X and Y to be complete seminormed spaces and (A_n) to be a sequence of linear operators from X into Y. We denote by $B(X,Y)$ the space of bounded linear operators on X into Y. Throughout the paper S denotes the unit ball in X, that is $S = \{x \in X : \|x\| \leq 1\}$ is the closed unit sphere in X.

The α and β-duals of Köthe have been generalized by Robinson [6] who replaced scalar sequences by sequences of linear operators. Accordingly, we define α and β duals of a subspace E of $s(X)$ by

$$E^\alpha = \{(A_n) : \sum_n \|A_n x_n\| \text{ converges for all } x = (x_n) \in E\},$$

$$E^\beta = \{(A_n) : \sum_n A_n x_n \text{ converges in Y, for all } x = (x_n) \in E\}.$$

Clearly $E^\alpha \subset E^\beta$ if Y is complete and the inclusion may be strict. X^* will denote the continuous dual of X, this is $B(X,C)$.

2. MAIN RESULTS

Before proving the main results we give some definitions. We consider a set D of sequences $d = (d_n)$ of non-negative real numbers with the following properties:

(i) For each positive integer n there exists $d \in D$ with $d_n > 0$,

(ii) D is directed in the sense that for $d, h \in D$ there exists $u \in D$ such that $u_n \geq d_n, h_n$ for all n.

For $d = (d_n) \in D$ and X a seminormed vector space, we define the following sequence spaces:

$$L_\infty(X,d) = \{x = (x_n) : D_d(x) = \sup_n \|x_n\| d_n < \infty, \quad x_n \in X \text{ for all } n, \quad d \in D\},$$
$$C_0(X,d) = \{x = (x_n) : \lim_n \|x_n\| d_n = 0, \quad x_n \in X \text{ for all } n, \quad d \in D\}.$$

PROPOSITION 2.1 $C_0(X,d)$ is a closed subspace of $L_\infty(X,d)$.

PROOF. Let $x \in \bar{C}_0(X,d)$ and $d = (d_n) \in D$. Given $\epsilon > 0$ there exists $x' = (x'_n) \in C_0(X,d)$ such that $D_d(x - x') < \frac{\epsilon}{2}$. If N is such that $d_n\|x'_n\| < \frac{\epsilon}{2}$ for $n \geq N$, then for $n \geq N$ we have

$$d_n\|x_n\| = d_n\|x_n - x'_n + x'_n\| \leq d_n(\|x_n - x'_n\| + \|x'_n\|) < \epsilon$$

which proves that $x \in C_0(X,d)$.

PROPOSITION 2.2 If X is complete then $C_0(X,d)$ and $L_\infty(X,d)$ are FK spaces.

PROOF. Let X be a complete seminormed space. We show that $L_\infty(X,d)$ is complete. Let $x = (x^i_n)$ be a Cauchy sequence in $L_\infty(X,d)$. Then $\|x^i_n - x^j_n\| \leq d_n^{-1} D_d(x^i - x^j)$ therefore (x^i_n) is Cauchy in X. Let $x_n = \lim_i x^i_n$. Now we will show that $x = (x_n) \in L_\infty(X,d)$ and $x^i \to x$. In fact, let $h \in D$ and $\epsilon > 0$. Choose N such that $D_h(x^i - x^j) < \epsilon$ if $i,j \geq N$. It follows from this that, we have $\|x^i_n - x_n\|h_n < \epsilon$ for all n and $i \geq N$. Let $H = D_h(x_N)$. If $\|x_n\| \leq \|x^N_n\|$ then $\|x_n\|h_n \leq H$. If $\|x_n\| > \|x^N_n\|$ then

$$\|x_n\| = \|x_n - x^N_n + x^N_n\|h_n \leq \|x_n - x^N_n\|h_n + \|x^N_n\|h_n < \epsilon + H$$

which shows that $L_\infty(X,d)$ is complete. The completeness of $C_0(X,d)$ follows from the completeness of $L_\infty(X,d)$ and the Proposition 2.1.

THEOREM 2.3 $C_0(X,d) = L_\infty(X,d)$ if and only if for each $d = (d_n) \in D$ there exists $h = (h_n) \in D$ and a sequence (u_n) of non-negative real numbers such that $u_n \to 0$ and $d_n \leq u_n h_n$ for all n.

PROOF. Let $x \in L_\infty(X,d)$. Given $d = (d_n) \in D$ there exist $h = (h_n) \in D$ and a sequence (u_n) of non-negative real numbers such that $u_n \to 0$ and $d_n \leq u_n h_n$ for all n. Now, for $x \in L_\infty(X,d)$, we have

$$d_n\|x_n\| \leq u_n h_n\|x_n\| \leq u_n D_h(x).$$

This concludes the proof of the theorem with the Proposition 2.1.

LEMMA 2.4 In order for $C_0(X,d) \subset C_0(X,h)$ it is necessary and sufficient that $\liminf_n \frac{d_n}{h_n} > 0$.

PROOF. Suppose that $\liminf_n \frac{d_n}{h_n} = \alpha > 0$. Then since $d_n > \alpha h_n$ the inclusion $C_0(X,d) \subset C_0(X,h)$ is obvious. Now we suppose $\liminf_n \frac{d_n}{h_n} = 0$. Then there exists a subsequence $(n(p))$ of (n) such that $h_{n(p)} > p d_{n(p)}$ for $p = 1,2\ldots$. Now define a sequence $x = (x_n)$ by putting $x_{n(p)} = v d^{-1}_{n(p)} p^{-1}$ for $p = 1,2,\ldots$and $x_n = \theta$ otherwise where $v \in X$ and $\|v\| = 1$. Then we have $x = (x_n) \in C_0(X,d)$ but $x \notin C_0(X,h)$ since $\|h_{n(p)} x_{n(p)}\| = \|h_{n(p)} d^{-1}_{n(p)} p^{-1} v\| > 1$. The concludes the proof of the theorem.

LEMMA 2.5 In order for $C_0(X,h) \subset C_0(X,d)$ it is necessary and sufficient that $\limsup_n \frac{d_n}{h_n} < \infty$.

PROOF. Suppose that $\limsup_n \frac{d_n}{h_n} < \infty$. Then there is $K > 0$ such that $d_n < K h_n$ for all large values of n. The inclusion $C_0(X,h) \subset C_0(X,d)$ is obvious. Now we suppose $\limsup_n \frac{d_n}{h_n} = \infty$. Then there exists a subsequence $(n(p))$ of (n) such that $d_{n(p)} > p h_{n(p)}$ for $p = 1,2,\ldots$. We define a sequence $x = (x_n)$ by putting $x_{n(p)} = v h^{-1}_{n(p)} p^{-1}$ for $p = 1,2,3,\ldots$and $x_n = \theta$ otherwise where $v \in X$ and $\|v\| = 1$. Then we have $x \in C_0(X,h)$ but $x \notin C_0(X,d)$ since $\|d_{n(p)} x_{n(p)}\| = \|d_{n(p)} h^{-1}_{n(p)} p^{-1} v\| > 1$. The concludes the proof of the lemma.

Combining Lemma 2.4. and 2.5. we have following theorem.

THEOREM 2.6 $C_0(X,h) = C_0(X,d)$ if and only if $0 < \liminf_n \frac{d_n}{h_n} \leq \limsup_n \frac{d_n}{h_n} < \infty$.

THEOREM 2.7 Let $\liminf_n \frac{d_n}{h_n} > 0$. The identity mapping of $C_0(X,d)$ into $C_0(X,h)$ is continuous.

PROOF. Let $\liminf_n \frac{d_n}{h_n} > 0$. Then $C_0(X,d) \subset C_0(X,h)$. There exists $\alpha > 0$ such that $d_n > \alpha h_n$ for all n. Thus for $x \in C_0(X,d)$ we have $\alpha D_h(x) \leq D_d(x)$ Hence the identity mapping is continuous.

3. GENERALIZED KÖTHE-TOEPLITZ DUALS

Now we determine Köthe-Toeplitz duals in the operator case for the sequence space $C_0(X,d)$. For the more interesting sequence spaces generalized Köthe-Toeplitz duals were determined by Maddox [3]. In the following theorems we suppose in general that (A_n) is a sequence of linear operators A_n mapping a complete seminormed space X into a complete seminormed space Y. Let $(A_n) = (A_1, A_2, \ldots)$ be a sequence in $B(X, Y)$. Then the group norm of (A_n) is defined by

$$\|(A_n)\| = \sup \|\sum_{n=1}^{k} A_n x_n\|$$

where the supremum is taken over all $k \in N$ and all $x_n \in S$. This argument was introduced by Robinson[6]. This concept was termed as group norm by Lorentz and Macphail [1]. We start with the proposition given by Maddox [3].

PROPOSITION [M][3] If (A_n) is a sequence in $B(X, Y)$ and we write $R_k = (A_k, A_{k+1}, \ldots)$ then $\|\sum_{n=k}^{k+p} A_n x_n\| \leq \|R_k\|.\max\{\|x_n\| : k \leq n \leq k+p\}$, for any x_n and all $k \in N$, and all $p > 0$ integers.

THEOREM 3.1 Let $(d_n) \in D$. Then $(A_n) \in C_0^{\alpha}(X, d)$ if and only if there exists an integer k such that

(i) $A_n \in B(X, Y)$ for each $n \geq k$ and

(ii) $\sum_{n \geq k} \|A_n\| d_n^{-1} < \infty$.

PROOF. For the sufficiency, let $x = (x_n) \in C_0(X, d)$ and (i), (ii) hold. Then there exists an integer n_1 such that $\|x_n\| d_n < 2\epsilon$ for all $n \geq n_1$ and there exists an integer $n_2 \geq k$ such that

$$\sum_{n \geq n_2} \|A_n\| d_n^{-1} < \frac{\epsilon}{2}$$

for a given $\epsilon > 0$. Put $H = \max(n_1, n_2)$ so that

$$\sum_{n \geq H} \|A_n x_n\| = \sum_{n \geq H} \|A_n\| \|x_n\| \leq \sum_{n \geq H} \|A_n\| 2\epsilon d_n^{-1} < \epsilon,$$

and therefore $(A_n) \in C_0^{\alpha}(X, d)$.

Conversely, suppose that $(A_n) \in C_0^{\alpha}(X, d)$. If (i) does not hold then there exists a strictly increasing sequence (n_i) of natural numbers such that A_{n_i} is not bounded for each i and a sequence (v_n) in S such that $\|A_{n_i} v_{n_i}\| > d_{n_i} i$, for each $i \geq 1$. Define the sequence $x = (x_n)$ by putting $x_{n_i} = v_{n_i} d_{n_i}^{-1} i^{-1}$ for each $i \geq 1$ and $x = \theta$ otherwise. We have $x \in C_0(X, d)$ but $\|A_{n_i} x_{n_i}\| > 1$ for each $i \geq 1$ and so $\sum_n \|A_n x_n\|$ diverges, which gives a contradiction.

Now we suppose $(A_n) \in C_0^{\alpha}(X, d)$ and $\sum_{n \geq k} \|A_n\| d_n^{-1} = \infty$. We choose $k = n_1 < n_2 < n_3 \ldots$ such that $\sum_{n=n_i}^{n_{i+1}-1} \|A_n\| d_n^{-1} > i$ for $i \in N$. Moreover for each $n \geq k$ there exists a sequence (v_n) in S such that $2\|A_n v_n\| \geq \|A_n\|$. Define the sequence $x = (x_n)$ by putting $x_n = v_n d_n^{-1} i^{-1}$ for $n_i \leq n \leq n_{i+1} - 1$ for $i = 1, 2, \ldots$ and $x_n = \theta$ otherwise so that $x \in C_0(X, d)$ since

$$\|x_n\| d_n = \frac{\|v_n\|}{i} \to 0 \text{ as } n \to \infty.$$

Then we have

$$\begin{aligned}
\sum_n \|A_n x_n\| &= \sum_{i=1}^{\infty} \sum_{n=n_i}^{n_{i+1}-1} \|A_n v_n d_n^{-1} i^{-1}\| \\
&\geq \frac{1}{2} \sum_{i=1}^{\infty} \sum_{n=n_i}^{n_{i+1}-1} \|A_n\| d_n^{-1} i^{-1} \\
&\geq \frac{1}{2} \sum_{i=1}^{\infty} 1
\end{aligned}$$

which contradicts our assumption that $\sum_n \|A_n x_n\| < \infty$. This completes the proof.

It is clear that the conditions of the theorem 3.1. are also necessary and sufficient for $(A_n) \in l_{\infty}^{\alpha}(X, d)$ then we have $C_0^{\alpha}(X, d) = l_{\infty}^{\alpha}(X, d)$.

COROLLARY 3.2 ([5],Theorem 1.) Let $p_n = O(1)$. Then $(A_n) \in C_0^{\alpha}(X,p)$ if and only if there exists an integer k such that condition (i) of Theorem 3.1. holds and

(iii)there exists an integer $N > 1$ such that $\sum_{n \geq k} \|A_n\| N^{-\frac{1}{p_n}} < \infty$.

COROLLARY 3.3([3],Proposition 3.4.) $(A_n) \in C_0^{\alpha}(X)$ if and only if there exists an integer k such that condition (i) of Theorem 3.1. holds and

(iv) $\sum_{n=k}^{\infty} \|A_n\| < \infty$.

THEOREM 3.4 Let $(d_n) \in D$. Then $(A_n) \in C_0^{\beta}(X,d)$ if and only if there exists an integer k such that condition (i) of Theorem 3.1. holds and

(v) $\|R_k(d)\| = \|(d_k^{-1} A_k, d_{k+1}^{-1} A_{k+1}, \ldots)\| < \infty$.

PROOF. For the sufficiency, let $(x_n) \in C_0(X,d)$ and choose $m_1 > m \geq k$. Then, by the proposition [M] we have for $m \geq k$

$$\| \sum_{n=m}^{m_1} A_n x_n \| = \| \sum_{n=m}^{m_1} d_n^{-1} A_n d_n x_n \| \leq \max\{d_n \|x_n\| : m \leq n \leq m_1\} \|R_k(d)\|.$$

That is $\sum_n A_n x_n$ is convergent in Y whence $(A_n) \in C_0^{\beta}(X,d)$. Conversely (i) can be proved in the way of Theorem 3.1. For the necessity of (v), suppose that $\|R_k(d)\| = \infty$ for all $n \geq k$ then there exists a strictly increasing sequence (n_i) of natural numbers such that $v_{n_i} \in S$ and $\| \sum_{n=n_i}^{n_{i+1}-1} d_n{}^{-1} A_n v_n \| > i$ for $i \in N$. Define the sequence $x = (x_n)$ by putting $x_n = v_n d_n^{-1} i^{-1}$ for $n_i \leq n \leq n_{i+1} - 1$, $i = 1, 2, \ldots$ and $x_n = \theta$ otherwise. We have $x \in C_0(X,d)$ but for each $i \geq 1$

$$\| \sum_{n=n_i}^{n_{i+1}-1} A_n x_n \| = \| \sum_{n=n_i}^{n_{i+1}-1} A_n v_n d_n^{-1} i^{-1} \| > 1$$

Therefore $\sum_n A_n x_n$ diverges, which gives a contradiction. This proves the theorem.

COROLLARY 3.5 ([3],Proposition 3.1.) $d_n = 1$ for all n, $(A_n) \in C_0^{\beta}(X)$ if and only if condition (i) of Theorem 3.1. holds and $\|R_k\| < \infty$.

THEOREM 3.6 $Y = C$ and $f_n \in X^*$ for $n \geq 1$ then $C_0^{\alpha}(X,d) = C_0^{\beta}(X,d) = M_0(X^*,d)$ where $M_0(X^*,d) = \{F = (f_n) : f_n \in X^*, \sum_n \|f_n\| d_n^{-1} < \infty\}$.

PROOF. We show that $C_0^{\beta}(X,d) \subset M_0(X^*,d)$, which is sufficient to prove of the theorem. We suppose $F \notin M_0(X^*,d)$ then there exists a strictly increasing sequence (n_i) and a sequence (v_n) in S such that $\|f_n\| < 2|f_n(v_n)|$ and $\sum_{n=n_i}^{n_{i+1}-1} \|f_n\| d_n^{-1} > i$ for $i \in N$. Define the sequence $x = (x_n)$ by putting $x_n = sgn(f_n(v_n)) d_n^{-1} i^{-1} v_n$ for $n_i \leq n \leq n_{i+1} - 1$, $i = 1, 2, \ldots$ and $x_n = \theta$ otherwise. Then $x \in C_0(X,d)$ but $\sum_n f_n(x_n) = \sum_{i=1}^{\infty} \sum_{n=n_i}^{n_{i+1}-1} f_n(x_n)$ diverges and so $F \notin C_0^{\beta}(X,d)$. Thus $C_0^{\beta}(X,d) \subset M_0(X^*,d)$ and the proof is complete.

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