ON X-VALUED SEQUENCE SPACES

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ABSTRACT. Certain spaces of X-valued sequences are introduced and some of their properties are investigated. Köthe-Toeplitz duals of these spaces are examined.

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1. INTRODUCTION AND BACKGROUND.

Let $c_0, c, l_0$ and $s$ respectively denote the spaces of null sequences, convergent sequences, bounded sequences and all sequences. Let $X$ be a complex linear space with zero element $0$ and $X, (X, ||.||)$ be a seminormed space. We may define $c_0(X)$ the null $X$-valued sequences, $c(X)$ the convergent $X$-valued sequences, $l_0(X)$ the bounded $X$-valued sequences and $s(X)$ the vector space of all $X$-valued sequences. If we take $X = C$ the set of complex numbers these spaces reduce to the already familiar spaces $c_0, c, l_0$ and $s$ respectively. These spaces of $X$-valued sequences have been studied by Maddox[2,3], Rath[5], Pehlivan[4] and others. We take $X$ and $Y$ to be complete seminormed spaces and $(A_n)$ to be a sequence of linear operators from $X$ into $Y$. We denote by $B(X, Y)$ the space of bounded linear operators on $X$ into $Y$. Throughout the paper $S$ denotes the unit ball in $X$, that is $S = \{ x \in X : ||x|| \leq 1 \}$ is the closed unit sphere in $X$.

The $\alpha$ and $\beta$-duals of Köthe have been generalized by Robinson [6] who replaced scalar sequences by sequences of linear operators. Accordingly, we define $\alpha$ and $\beta$ duals of a subspace $E$ of $s(X)$ by

$$E^\alpha = \{ (A_n) : \sum_n ||A_n x_n|| \text{ converges for all } x = (x_n) \in E \},$$

$$E^\beta = \{ (A_n) : \sum_n A_n x_n \text{ converges in } Y \text{ for all } x = (x_n) \in E \}.$$

Clearly $E^\alpha \subset E^\beta$ if $Y$ is complete and the inclusion may be strict. $X^*$ will denote the continuous dual of $X$, this is $B(X, C)$.

2. MAIN RESULTS

Before proving the main results we give some definitions. We consider a set $D$ of sequences $d = (d_n)$ of non-negative real numbers with the following properties:

(i) For each positive integer $n$ there exists $d \in D$ with $d_n > 0$,

(ii) $D$ is directed in the sense that for $d, h \in D$ there exists $u \in D$ such that $u_n \geq d_n, h_n$ for all $n$.

For $d = (d_n) \in D$ and $X$ a seminormed vector space, we define the following sequence spaces:

$$L_\infty(X, d) = \{ x = (x_n) : D_d(x) = \sup_n ||x_n||d_n < \infty, \ x_n \in X \text{ for all } n, \ d \in D \},$$

$$C_0(X, d) = \{ x = (x_n) : \lim_n ||x_n||d'_n = 0, \ x_n \in X \text{ for all } n, \ d \in D \}.$$

PROPOSITION 2.1 $C_0(X, d)$ is a closed subspace of $L_\infty(X, d)$. 


PROOF. Let \( x \in \tilde{C}_0(X,d) \) and \( d = (d_n) \in D \). Given \( \epsilon > 0 \) there exists \( x' = (x'_n) \in C_0(X,d) \) such that \( D_d(x - x') < \frac{\epsilon}{2} \). If \( N \) is such that \( d_n ||x'_n|| < \frac{\epsilon}{2} \) for \( n \geq N \), then for \( n \geq N \) we have
\[
d_n ||x_n - x'_n|| = d_n ||x_n - x'_n + x'_n|| \leq d_n (||x_n - x'_n|| + ||x'_n||) < \epsilon
\]
which proves that \( x \in C_0(X,d) \).

PROPOSITION 2.2 If \( X \) is complete then \( C_0(X,d) \) and \( L_\infty(X,d) \) are FK spaces.

PROOF. Let \( X \) be a complete seminormed space. We show that \( L_\infty(X,d) \) is complete. Let \( x = (x_n) \) be a Cauchy sequence in \( L_\infty(X,d) \). Then \( ||x_n - x'_n|| \leq d_n^{-1} D_d(x - x') < \epsilon \), therefore \( (x_n) \) is Cauchy in \( X \). Let \( x_n = \lim_n x_n \). Now we will show that \( x = (x_n) \in L_\infty(X,d) \) and \( x - x \). In fact, let \( h \in D \) and \( \epsilon > 0 \). Choose \( N \) such that \( D_h(x' - x') < \epsilon \) if \( i,j \geq N \). It follows from this that, we have \( ||x_n - x||_h < \epsilon \) for all \( n \) and \( i \geq N \). Let \( H = D_h(x_N) \). If \( ||x_n|| \leq ||x_N|| \) then \( ||x_n||_h \leq H \). If \( ||x_n|| > ||x_N|| \) then
\[
||x_n|| = ||x_n - x_N + x_N||_h \leq ||x_n - x_N||_h + ||x_N||_h < \epsilon + H
\]
which shows that \( L_\infty(X,d) \) is complete. The completeness of \( C_0(X,d) \) follows from the completeness of \( L_\infty(X,d) \) and the Proposition 2.1.

THEOREM 2.3 \( C_0(X,d) \subseteq L_\infty(X,d) \) if and only if for each \( d = (d_n) \in D \) there exists \( h \in D \) and a sequence \( (u_n) \) of non-negative real numbers such that \( u_n \rightarrow 0 \) and \( d_n \leq u_n h_n \) for all \( n \).

PROOF. Let \( x \in L_\infty(X,d) \). Given \( d = (d_n) \in D \) there exist \( h = (h_n) \in D \) and a sequence \( (u_n) \) of non-negative real numbers such that \( u_n \rightarrow 0 \) and \( d_n \leq u_n h_n \) for all \( n \). Now, for \( x \in L_\infty(X,d) \), we have
\[
d_n ||x_n|| \leq u_n h_n ||x_n|| \leq u_n D_h(x)
\]
This concludes the proof of the theorem with the Proposition 2.1.

LEMMA 2.4 In order for \( C_0(X,d) \subseteq C_0(X,h) \) it is necessary and sufficient that \( \liminf_n \frac{h_n}{d_n} > 0 \).

PROOF. Suppose that \( \liminf_n \frac{h_n}{d_n} = \alpha > 0 \). Then since \( d_n > \alpha h_n \) the inclusion \( C_0(X,d) \subset C_0(X,h) \) is obvious. Now we suppose \( \liminf_n \frac{h_n}{d_n} = 0 \). Then there exists a subsequence \( (n(p)) \) of \( (n) \) such that \( h_{n(p)} > pd_{n(p)} \) for \( p = 1,2,\ldots \). Now define a sequence \( x = (x_n) \) by putting \( x_{n(p)} = ud_{n(p)}^{-1}p^{-1} \) for \( p = 1,2,\ldots \) and \( x_n = \theta \) otherwise where \( v \in X \) and \( ||v|| = 1 \). Then we have \( x = (x_n) \in C_0(X,d) \) but \( x \notin C_0(X,h) \) since \( ||h_{n(p)} x_{n(p)}|| = ||h_{n(p)} d_{n(p)}^{-1} p^{-1} v|| > 1 \). The concludes the proof of the theorem.

LEMMA 2.5 In order for \( C_0(X,h) \subset C_0(X,d) \) it is necessary and sufficient that \( \limsup_n \frac{d_n}{h_n} < \infty \).

PROOF. Suppose that \( \limsup_n \frac{d_n}{h_n} = \beta < \infty \). Then there is \( K > 0 \) such that \( d_n < Kh_n \) for all large values of \( n \). The inclusion \( C_0(X,h) \subset C_0(X,d) \) is obvious. Now we suppose \( \limsup_n \frac{d_n}{h_n} = \infty \). Then there exists a subsequence \( (n(p)) \) of \( (n) \) such that \( d_{n(p)} > ph_{n(p)} \) for \( p = 1,2,\ldots \). We define a sequence \( x = (x_n) \) by putting \( x_{n(p)} = \theta \) otherwise where \( v \in X \) and \( ||v|| = 1 \). Then we have \( x \in C_0(X,h) \) but \( x \notin C_0(X,d) \) since \( ||d_{n(p)} x_{n(p)}|| = ||d_{n(p)} h_{n(p)}^{-1} p^{-1} v|| > 1 \). The concludes the proof of the lemma.

Combining Lemma 2.4. and 2.5. we have following theorem.

THEOREM 2.6 \( C_0(X,h) = C_0(X,d) \) if and only if \( \liminf_n \frac{h_n}{d_n} \leq \limsup_n \frac{d_n}{h_n} < \infty \).

THEOREM 2.7 Let \( \liminf_n \frac{d_n}{h_n} > 0 \). The identity mapping of \( C_0(X,d) \) into \( C_0(X,h) \) is continuous.

PROOF. Let \( \liminf_n \frac{d_n}{h_n} > 0 \). Then \( C_0(X,d) \subset C_0(X,h) \). There exists \( \alpha > 0 \) such that \( d_n > \alpha h_n \) for all \( n \). Thus for \( x \in C_0(X,d) \) we have \( \alpha D_h(x) \leq D_d(x) \) hence the identity mapping is continuous.

3. GENERALIZED KÖTHE-TOEPLITZ DUALS

Now we determine Köthe-Toeplitz duals in the operator case for the sequence space \( C_0(X,d) \). For the more interesting sequence spaces generalized Köthe-Toeplitz duals were determined by Maddox [3]. In the following theorems we suppose in general that \( (A_n) \) is a sequence of linear operators \( A_n \) mapping
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a complete seminormed space $X$ into a complete seminormed space $Y$. Let $(A_n) = (A_1, A_2, \ldots)$ be a sequence in $B(X, Y)$. Then the group norm of $(A_n)$ is defined by

$$
\|(A_n)\| = \sup \| \sum_{n=1}^{k} A_n x_n \|
$$

where the supremum is taken over all $k \in \mathbb{N}$ and all $x_n \in S$. This argument was introduced by Robinson [6]. This concept was termed as group norm by Lorentz and Macphail [1]. We start with the proposition given by Maddox [3].

**PROPOSITION [M][3]** If $(A_n)$ is a sequence in $B(X, Y)$ and we write $R_k = (A_k, A_{k+1}, \ldots)$ then $\| \sum_{n=k}^{p} A_n x_n \| \leq \| R_k \| \max \{ \| x_n \| : k \leq n \leq k+p \}$, for any $x_n$ and all $k \in \mathbb{N}$, and all $p > 0$ integers.

**THEOREM 3.1** Let $(d_n) \in D$. Then $(A_n) \in C^0(X, d)$ if and only if there exists an integer $k$ such that

(i) $A_n \in B(X, Y)$ for each $n \geq k$ and

(ii) $\sum_{n=k}^{\infty} \| A_n \| d_n^{-1} < \infty$.

**PROOF.** For the sufficiency, let $x = (x_n) \in C_0(X, d)$ and (i), (ii) hold. Then there exists an integer $n_1$ such that $\| x_n \| d_n < 2\epsilon$ for all $n \geq n_1$ and there exists an integer $n_2 \geq k$ such that

$$
\sum_{n \geq n_2} \| A_n \| d_n^{-1} < \frac{\epsilon}{2}
$$

for a given $\epsilon > 0$. Put $H = \max(n_1, n_2)$ so that

$$
\sum_{n \geq H} \| A_n x_n \| = \sum_{n \geq H} \| A_n \| \| x_n \| \leq \sum_{n \geq H} \| A_n \| 2\epsilon d_n^{-1} < \epsilon,
$$

and therefore $(A_n) \in C^0(X, d)$.

Conversely, suppose that $(A_n) \in C^0(X, d)$. If (i) does not hold then there exists a strictly increasing sequence $(n_i)$ of natural numbers such that $A_{n_i}$ is not bounded for each $i$ and a sequence $(v_n)$ in $S$ such that $\| A_{n_i} v_n \| > d_{n_i}$ for each $i \geq 1$. Define the sequence $x = (x_n)$ by putting $x_{n_i} = v_i d_{n_i}^{-1}$ for each $i \geq 1$ and $x = \theta$ otherwise. We have $x \in C_0(X, d)$ but $\| A_{n_i} x_n \| > 1$ for each $i \geq 1$ and so $\sum_n \| A_n x_n \|$ diverges, which gives a contradiction.

Now we suppose $(A_n) \in C^0(X, d)$ and $\sum_{n \geq k} \| A_n \| d_n^{-1} = \infty$. We choose $k = n_1 < n_2 < n_3 \ldots$ such that $\sum_{n=n_i}^{n_{i+1}-1} \| A_n \| d_n^{-1} > 1$ for $i \in \mathbb{N}$. Moreover for each $n \geq k$ there exists a sequence $(v_n)$ in $S$ such that $2\| A_n v_n \| \geq \| A_n \|$. Define the sequence $x = (x_n)$ by putting $x_n = v_n d_n^{-1} i^{-1}$ for $i \leq n \leq n_{i+1} - 1$ for $i = 1, 2, \ldots$ and $x_n = \theta$ otherwise so that $x \in C_0(X, d)$ since

$$
\| x_n \| d_n = \frac{\| v_n \|}{i} \to 0 \text{ as } n \to \infty.
$$

Then we have

$$
\sum_n \| A_n x_n \| = \sum_{i=1}^{\infty} \sum_{n=n_i}^{n_{i+1}-1} \| A_n v_n d_n^{-1} i^{-1} \| \geq \frac{1}{2} \sum_{i=1}^{\infty} \sum_{n=n_i}^{n_{i+1}-1} \| A_n \| d_n^{-1} i^{-1} \geq \frac{1}{2} \sum_{i=1}^{\infty} 1
$$

which contradicts our assumption that $\sum_n \| A_n x_n \| < \infty$. This completes the proof.

It is clear that the conditions of the theorem 3.1. are also necessary and sufficient for $(A_n) \in l^\infty(X, d)$ then we have $C^0_0(X, d) = l^\infty_0(X, d)$. 

COROLLARY 3.2 ([5], Theorem 1.) Let \( p_n = O(1) \). Then \( (A_n) \in C_0^0(X, p) \) if and only if there exists an integer \( k \) such that condition (i) of Theorem 3.1 holds and

(iii) there exists an integer \( N > 1 \) such that \( \sum_{n \geq k} \|A_n\| N^{-\frac{1}{n}} < \infty \).

COROLLARY 3.3 ([3], Proposition 3.4.) \( (A_n) \in C_0^0(X) \) if and only if there exists an integer \( k \) such that condition (i) of Theorem 3.1 holds and

(iv) \( \sum_{n=m}^{\infty} \|A_n\| < \infty \).

THEOREM 3.4 Let \( (d_n) \in D \). Then \( (A_n) \in C_0^0(X, d) \) if and only if there exists an integer \( k \) such that condition (i) of Theorem 3.1 holds and

(v) \( \|R_k(d)\| = \|(d_k^{-1}A_k, d_{k+1}^{-1}A_{k+1}, \ldots)\| < \infty \).

PROOF. For the sufficiency, let \( (x_n) \in C_0(X, d) \) and choose \( m_1 > m \geq k \). Then, by the proposition [M] we have for \( m \geq k \)

\[
\| \sum_{n=m_1}^{m_1} A_n x_n \| = \| \sum_{n=m_1}^{m_1} d_n^{-1} A_n d_n x_n \| \leq \max\{d_n \|x_n\| : m \leq n \leq m_1\}\|R_k(d)\|.
\]

That is \( \sum_n A_n x_n \) converges in \( Y \) whence \( (A_n) \in C_0^0(X, d) \). Conversely (i) can be proved in the way of Theorem 3.1. For the necessity of (v), suppose that \( \|R_k(d)\| = \infty \) for all \( n \geq k \) then there exists a strictly increasing sequence \( (n_i) \) of natural numbers such that \( v_{n_i} \in S \) and \( \sum_{n=n_i}^{n_{i+1}-1} d_n^{-1} A_n v_n \rangle > i \) for \( i \in N \). Define the sequence \( x = (x_n) \) by putting \( x_n = v_n d_n^{-1} - 1 \) for \( n \leq n \leq n_{i+1} - 1, \ i = 1, 2, \ldots \) and \( x_n = \theta \) otherwise. We have \( x \in C_0(X, d) \) but for each \( i \geq 1 \)

\[
\| \sum_{n=n_i}^{n_{i+1}-1} A_n x_n \| = \| \sum_{n=n_i}^{n_{i+1}-1} A_n v_n d_n^{-1} - 1 \| > 1
\]

Therefore \( \sum_n A_n x_n \) diverges, which gives a contradiction. This proves the theorem.

COROLLARY 3.5 ([3], Proposition 3.1.) \( x_n \) for all \( n, (A_n) \in C_0^0(X) \) if and only if condition (i) of Theorem 3.1 holds and \( \|R_k\| < \infty \).

THEOREM 3.6 \( Y = C \) and \( f_n \in X^* \) for \( n \geq 1 \) then \( C_0^0(X^*, d) = C_0^0(X, d) = M_0(X^*, d) \) where \( M_0(X^*, d) = \{ F = (f_n) : f_n \in X^*, \sum_{n=1}^{\infty} \|f_n\| d_n^{-1} < \infty \} \).

PROOF. We show that \( C_0^0(X^*, d) \subseteq M_0(X^*, d) \), which is sufficient to prove of the theorem. We suppose \( F \notin M_0(X^*, d) \) then there exists a strictly increasing sequence \( (n_i) \) and a sequence \( (v_n) \) in \( S \) such that \( \|f_n\| < 2\|f_n(v_n)\| \) and \( \sum_{n=n_i}^{n_{i+1}-1} \|f_n\| d_n^{-1} > i \) for \( i \in N \). Define the sequence \( x = (x_n) \) by putting \( x_n = \text{sgn}(f_n(v_n)) d_n^{-1} - 1 \) for \( n \leq n \leq n_{i+1} - 1, \ i = 1, 2, \ldots \) and \( x_n = \theta \) otherwise. Then \( x \in C_0(X, d) \) but \( \sum_n f_n(x_n) = \sum_{n=n_i}^{n_{i+1}-1} f_n(x_n) \) diverges and so \( F \notin C_0^0(X, d) \). Thus \( C_0^0(X, d) \subseteq M_0(X^*, d) \) and the proof is complete.

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