ASYMPTOTIC EQUIVALENCE OF SEQUENCES AND SUMMABILITY

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ABSTRACT: For a sequence-to-sequence transformation \( A \), let \( R_m A x = \sum_{n \geq m} |(Ax)_n| \) and \( \mu_m A x = \sup_{n \geq m} |(Ax)_n| \). The purpose of this paper is to study the relationship between the asymptotic equivalence of two sequences \((\lim_n x_n/y_n = 1)\) and the variations of asymptotic equivalence based on the ratios \( R_m A x / R_m A y \) and \( \mu_m A x / \mu_m A y \).

KEY WORDS: Asymptotically regular, Asymptotic equivalence.

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1. INTRODUCTION.

Let \( x = (x)_n \) and \( y = (y)_n \) be infinite sequences, and let \( A \) be a sequence-to-sequence transformation. We write \( x \sim y \) if \( \lim_n x_n/y_n = 1 \). In order to compare rates of convergence of sequences, in [2] Pobyvanets introduced the concept of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative sequences, that is \( x \sim y \) implies \( A x \sim A y \). Furthermore, in [1] Fridy introduced new ways to compare rates by using the ratios \( R_m x / R_m y \), \( \mu_m x / \mu_m y \) when they tend to zero. In [2] Marouf studied the relationship of these ratios when they have limit one. In the present study we investigate some further properties involved with the ratios such \( \mu A x / \mu A y \), \( RA x / RA y \) when they have limit one.

2. NOTATIONS AND BASIC THEOREMS.

For a summability transformation \( A \), we use \( DA \) to denote the domain of \( A \):

\[
DA = \{ x : \sum_{k=0}^{\infty} a_k x_k \text{ converges for such } n \geq 0 \}
\]

and \( CA \) to denote the summability field:

\[
CA = \{ x : x \in DA, \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n x_k \text{ converges} \}
\]

Also

\[
P_\delta = \{ x : x_n \geq \delta > 0 \text{ for all } n \}
\]

and

\[
P = \{ x : x_n > 0 \text{ for all } n \}
\]

For a sequence \( x \) in \( \ell^1 \) or \( \ell^\infty \), we also define \( R_m x = \sum_{n \geq m} |x_n| \) and \( \mu_m x = \sup_{n \geq m} |x_n| \) for \( m \geq 0 \).

We list the following results without proof.
THEOREM 1. (Pobyvanets [2]). A nonnegative matrix $A$ is asymptotically regular if and only if for each fixed integer $m$, $\lim_{n \to \infty} a_{nm}/\sum_{k=0}^{\infty} a_{nk} = 0$.

THEOREM 2. A matrix $A$ is a $c_0 - c_0$ matrix (i.e. A preserves zero limits) if and only if:
(a) $\lim_{n \to \infty} a_{nk} = 0$ for $k = 0, 1, 2, \ldots$.
(b) There exists a number $M > 0$ such that for each $n$, $\sum_{k=0}^{\infty} |a_{nk}| < M$.

3. ASYMPTOTIC EQUIVALENCE PROPERTIES.

THEOREM 3. Let $A$ be a nonnegative matrix. Suppose $x \sim y$, and $x, y \in P_\delta$ for some $\delta > 0$. Then $\mu Ax \sim \mu Ay$ if and only if for each $i = 0, 1, 2, \ldots$

$$\lim_{n \to \infty} a_{ni}/\sum_{j=0}^{\infty} a_{nj} = 0.$$ 

PROOF. If $\lim_{n \to \infty} a_{ni}/\sum_{j=0}^{\infty} a_{nj} = 0$, $i = 0, 1, 2, \ldots$, we want to prove that $\mu Ax \sim \mu Ay$.

Since $x \sim y$, there exists a null sequence $\zeta$, such that

$$x_n = y_n(1 + \zeta_n), n = 0, 1, 2, \ldots;$$

then

$$\frac{(\mu Ax)_n}{(\mu Ay)_n} = \sup_{k \geq n} (Ax)_k \sup_{k \geq n} (Ay)_k$$

$$= \sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} x_i \sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} y_i$$

$$= \sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} (y_i + y_i \zeta_i) \sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} y_i$$

$$\leq 1 + \frac{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} y_i |\zeta_i|}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} y_i} + \frac{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} y_i |\zeta_i|}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} y_i}$$

where $N$ is a positive integer.

Since $\zeta$ is a null sequence, $\sup_j |\zeta_j| < \infty$, and for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that if $i \geq N$, then $|\zeta_i| < \epsilon$. Hence

$$\frac{(\mu Ax)_n}{(\mu Ay)_n} \leq 1 + \sup_j |\zeta_j| \sum_{i=0}^{N} \frac{\sup_{k \geq n} a_{ki} y_i}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} y_i} + \frac{\epsilon \sup_{k \geq N} \sum_{i=0}^{\infty} a_{ki} y_i}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} y_i}$$

$$\leq 1 + \sup_j |\zeta_j| \sum_{i=0}^{N} \frac{y_i \sup_{k \geq n} a_{ki}}{\delta \sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki}} + \epsilon$$

$$\leq 1 + \sup_j |\zeta_j| \sup_{0 \leq j \leq N} y_j \sum_{i=0}^{N} \frac{a_{ki}}{\sum_{i=0}^{\infty} a_{ki}} + \epsilon.$$ 

According to the hypothesis, there exists $N_1 \in \mathbb{N}$, such that if $k \geq N_1$, then $a_{ki}/\sum_{i=0}^{\infty} a_{ki} < \epsilon/N \sup_j |\zeta_j| \sup_{0 \leq j \leq N} y_j$. So if $n \geq N$, we have
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\[
\left(\frac{\mu Ax}{\mu Ay}\right)_n \leq 1 + \epsilon + \epsilon.
\]

This implies that \(\lim_{n \to \infty} \left(\frac{\mu Ax}{\mu Ay}\right)_n \leq 1\). Similarly, we may prove \(\lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki}}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki}} \leq 1\) and the two inequalities yield \(\lim_{n \to \infty} \left(\frac{\mu Ax}{\mu Ay}\right)_n = 1\).

Next, suppose \(\mu Ax \sim \mu Ay\) for any \(z \sim y\) such that \(x, y \in P_\delta\) for some \(\delta > 0\). We take \(x = y = (1,1,\ldots)\). Then \(\mu Ax \sim \mu Ay\), i.e.,

\[
\lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki}}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki}} = 1.
\]

Hence, there exists \(M > 0\), such that \(\{\sum_{i=0}^{\infty} a_{ki}\}_{k=0}^{\infty}\) is bounded by \(M\).

If \(\lim_{n \to \infty} a_{ni}/\sum_{j=0}^{\infty} a_{nj} \neq 0\) for some \(i\). Then there exists \(\lambda > 0\) and a sequence \(n_1 < n_2 < \ldots\), such that \(a_{nu}/\sum_{j=0}^{\infty} a_{nj} \geq \lambda, u = 1,2,3,\ldots\). Take \(t > 0\), and define \(x\) and \(y\) by

\[
y_n = 1, n = 0,1,2,\ldots
\]

and

\[
x_n = \begin{cases} 
1 & \text{if } n \neq i \\
1 + t & \text{if } n = i
\end{cases}
\]

It is clear that \(x \sim y\) and \(x, y \in P_1\). Consider the following limit:

\[
\lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ni}x_i}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ni}y_j} = \lim_{n \to \infty} \frac{\sup_{k \geq n} \left(\sum_{i=0}^{\infty} a_{ni} + t a_{ni}\right)}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ni}} \geq \lim_{n \to \infty} \frac{\sup_{k \geq n} \left(\sum_{i=0}^{\infty} a_{ni} + t \sum_{i=0}^{\infty} a_{ni}\right)}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ni}} = 1 + t\lambda.
\]

We can choose \(t = 1/\lambda\), which gives

\[
\lim_{n \to \infty} \left(\frac{\mu Ax}{\mu Ay}\right)_n = 2.
\]

This is a contradiction of \(\mu Ax \sim \mu Ay\).

**THEOREM 4.** Suppose \(A\) is a nonnegative matrix; then \(\mu x \sim \mu y\) implies \(\mu Ax \sim \mu Ay\) for any bounded sequences \(x, y \in P_\delta\), for some \(\delta > 0\), if and only if \(A\) satisfies the following three conditions:

(i) \(\left(\sum_{j=0}^{\infty} a_{ij}\right)_{i=0}^{\infty}\) is a bounded sequence dominated by some \(B\);

(ii) For any \(j = 0,1,2,\ldots\)

\[
\lim_{n \to \infty} \frac{\sup_{k \geq n} a_{kj}}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki}} = 0;
\]

(iii) For any infinite sequence \(j_1 < j_2 < j_3\ldots\)
Before we prove this theorem, we shall give some examples of $A$ which satisfy the above conditions (i), (ii), and (iii).

Example 1. $A = I$.

Example 2.

PROOF OF THEOREM 4. First, assume that for any bounded sequences $x, y \in P_1$, for some $\delta > 0$, if $x \sim y$ implies $\mu x \sim \mu y$; we wish to prove that $A$ satisfies the conditions (i), (ii) and (iii). Take $x = y = (1, 1, \ldots)$; then $x, y$ are bounded, $x, y \in P_1$, and $\mu x \sim \mu y$; so $\mu x \sim \mu y$. But $(\mu x)_n = \sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki}$. Hence, $(\sum_{i=0}^{\infty} a_{ki})_n$ should be bounded. This proves (i). To prove (ii) suppose there is a $j$ such that

\[
\sup_{k \geq n} \frac{\sum_{i=0}^{\infty} a_{kj}}{\sum_{i=0}^{\infty} a_{ki}} = \lambda
\]

for some $\lambda > 0$. As in the proof of Theorem 3, take $t > 0$ and define $y = (1, 1, \ldots)$ and

\[
x_n = \begin{cases} 
1 & \text{if } n \neq j, \\
1 + t & \text{if } n = j.
\end{cases}
\]

Then $x, y \in P_1$, $x, y$ are bounded, and $\mu x \sim \mu y$; so we have $\mu x \sim \mu y$. But

\[
= \lim_{n \to \infty} \frac{\sum_{i=0}^{\infty} a_{ki} x_i}{\sum_{i=0}^{\infty} a_{ki} y_i}
= \lim_{n \to \infty} \frac{t \sum_{i=0}^{\infty} a_{ki}}{\sum_{i=0}^{\infty} a_{ki}} - 1
= t\lambda - 1.
\]

By choosing $t = \frac{2}{\lambda}$, we get

\[
\lim_{n \to \infty} \sup_{k \geq n} \frac{\sum_{i=0}^{\infty} a_{ki} x_i}{\sum_{i=0}^{\infty} a_{ki} x_i} \geq 3 - 1 = 2.
\]

This is a contradiction $\mu x \sim \mu y$, so (ii) must hold.

Finally, we are going to prove (iii). For any given infinite sequence $j_1 < j_2 < \ldots$, we define $x$ and $y$ by

\[
y_n = 2 \text{ for every } n,
\]

and

\[
x_n = \begin{cases} 
2, & \text{if } n = j_u \text{ for } u = 1, 2, \ldots, \\
1, & \text{otherwise}.
\end{cases}
\]
It is easy to see that $x, y$ are bounded, $x, y \in P_1$ and $\mu x \sim \mu y$. This implies $\mu Ax \sim \mu Ay$.

Hence we have

$$1 = \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}x_j}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}y_j} = \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j \in J} a_{kj}x_j + \sum_{j \in J} a_{kj}x_j}{2 \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}}$$

where $J = \{j_1, j_2, j_3, \ldots\}$

$$= \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j \in J} a_{kj}x_j + \sum_{j \in J} a_{kj}\cdot \frac{1}{2}}{2 \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} \leq \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j \in J} a_{kj} + \sup_{k \geq n} \sum_{j \in J} a_{kj}}{2 \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} \leq \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j \in J} a_{kj} + \sup_{k \geq n} \sum_{j \in J} a_{kj}}{2 \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} + \frac{1}{2}.$$

Hence

$$1 \leq \frac{1 + \lim \sup_{k \geq n} \sum_{j \in J} a_{kj} + \frac{1}{2}}{2 \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}}.$$

This implies

$$\lim \sup_{k \geq n} \sum_{j \in J} a_{kj} \geq 1.$$

On the other hand, it is clear that

$$\lim \sup_{k \geq n} \sum_{j \in J} a_{kj} \leq 1.$$

Combining the last two inequalities together, we get

$$\lim \sup_{k \geq n} \sum_{j \in J} a_{kj} = 1,$$

which proves (iii).

Conversely, assume $A$ satisfies the conditions (i), (ii) and (iii), and suppose $x, y$ are bounded by some $M > 0$, $x, y \in P_1$ for some $\delta > 0$, and $\mu x \sim \mu y$. For any $\epsilon > 0$, since $x, y$ are bounded, there exists $N_1 \in \mathbb{N}$ such that if $j \geq N_1$, then

$$y_j \leq \lim \sup_{k \geq n} y_j + \epsilon$$

and also there exists an infinite sequence $j_1 < j_2 < \ldots$, such that

$$x_{j_i} \geq \lim \sup_{k \geq n} x_{j_i} - \epsilon$$

for $i = 1, 2, 3, \ldots$. Therefore

$$\lim \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}x_j = \lim \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}y_j.$$
$$\geq \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} x_{k}}{\sup_{k \geq n} \sum_{j=0}^{N_{1}} a_{kj} y_{j} + \sum_{j=N_{1}+1}^{\infty} a_{kj} y_{j}}$$

$$\geq \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} (\lim_{z \to \infty} \sup_{k \geq z} x_{z} - \varepsilon)}{M \sup_{k \geq n} \sum_{j=0}^{N_{1}} a_{kj} + \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} (\lim_{z \to \infty} \sup_{k \geq z} y_{z} + \varepsilon)}$$

$$\geq \lim_{n \to \infty} \frac{(\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}) \lim_{t \to \infty} \sup_{k \geq t} x_{t} - \varepsilon \sup_{k \geq n} \sum_{j=1}^{\infty} a_{kj}}{M \sup_{k \geq n} \sum_{j=0}^{N_{1}} a_{kj} + \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} + \sup_{k \geq n} (\sum_{j=N_{1}+1}^{\infty} a_{kj}) \lim_{t \to \infty} \sup_{k \geq t} y_{t}}$$

$$\leq \lim_{n \to \infty} \frac{(\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}) \lim_{t \to \infty} \sup_{k \geq t} x_{t}}{M \sup_{k \geq n} \sum_{j=0}^{N_{1}} a_{kj} + \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} + (\sup_{k \geq n} (\sum_{j=N_{1}+1}^{\infty} a_{kj}) \lim_{t \to \infty} \sup_{k \geq t} y_{t}}$$

$$- \frac{\varepsilon}{\delta}$$

(here, we used (iii) to deduce that

$$\lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} x_{k}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} y_{j}} = \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} x_{k}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} y_{j}} = 1 = 1$$)

$$\geq \lim_{n \to \infty} \frac{1}{B_{1} + B_{2} + B_{3}} - \frac{\varepsilon}{\delta}$$

$$\geq \lim_{n \to \infty} \frac{1}{\sup_{k \geq n} \sum_{j=0}^{N_{1}} a_{kj} + \sup_{k \geq n} \sum_{j=N_{1}+1}^{\infty} a_{kj} + (\sup_{k \geq n} \sum_{j=N_{1}+1}^{\infty} a_{kj} \lim_{t \to \infty} \sup_{k \geq t} x_{t}}$$

where

$$B_{1} = \frac{M \sup_{k \geq n} \sum_{j=0}^{N_{1}} a_{kj} \lim_{t \to \infty} \sup_{k \geq t} x_{t}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} \lim_{t \to \infty} \sup_{k \geq t} x_{t}}$$

$$B_{2} = \frac{\varepsilon \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} \lim_{t \to \infty} \sup_{k \geq t} x_{t}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} \lim_{t \to \infty} \sup_{k \geq t} x_{t}}$$

$$B_{3} = \frac{(\sup_{k \geq n} \sum_{j=0}^{N_{1}+1} a_{kj} \lim_{t \to \infty} \sup_{k \geq t} x_{t}}}{\sup_{k \geq n} \sum_{j=1}^{\infty} a_{kj} \lim_{t \to \infty} \sup_{k \geq t} x_{t}}$$

For the fixed $N_{1}$, combining conditions (ii) and (iii), we can easily prove

$$\frac{\sup_{k \geq n} \sum_{j=0}^{N_{1}} a_{kj}}{\sup_{k \geq n} \sum_{j=1}^{\infty} a_{kj}} \to 0 \text{ as } n \to \infty.$$
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\[
\frac{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}}{\sup_{k \geq n} \sum_{i=1}^{\infty} a_{ki}} < 1 + \varepsilon \quad \text{(by (iii))},
\]

and
\[
\frac{\sup_{k \geq n} \sum_{i=1}^{\infty} a_{ki}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} < 1 + \varepsilon \quad \text{(by (iii))}.
\]

These imply that if \( n \geq N_2 \)
\[
\frac{1}{M \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} \geq \frac{1}{\varepsilon} + \frac{1}{\delta} + \varepsilon.
\]

Hence
\[
\lim_{n \to \infty} \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} x_j = \frac{1}{\varepsilon} + \frac{1}{\delta} + \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we have
\[
\lim_{n \to \infty} \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} x_j \geq 1.
\]

Similarly, we can prove
\[
\lim_{n \to \infty} \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} y_j \leq 1.
\]

Thus, we have finished the proof.

**REMARK.**

Let \( A \) be a nonnegative matrix, \( A = (a_{ij}) \). If \( A \) satisfies the following two conditions, then \( A \) satisfies the conditions (i), (ii), (iii) of theorem 4:

a) There exists \( \lambda > 0 \), such that
\[
\lim_{n \to \infty} a_{nn} = \lambda
\]

b) \( \lim_{n \to \infty} \sum_{j \neq n} a_{nj} = 0 \)

**PROOF OF THE REMARK.** If \( A \) satisfies the above conditions a and b, it is easy to see that \( A \) satisfies (i) in theorem 4. To prove (iii), let \( j_1, j_2, \ldots \) be an infinity sequence: \( j_1 < j_2 < \ldots \) Then
\[
\lim_{n \to \infty} \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} = \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} = \frac{\lambda}{\lambda + 0} = 1
\]

This inequality gives that
\[
\lim_{n \to \infty} \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} = 1
\]

Next, let's prove (ii) of theorem 4. In fact, for any fixed \( j = 0, 1, 2, \ldots \)
\[
\lim_{n \to \infty} \frac{\sup_{k \geq n} a_{kj}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} \leq \frac{\sup_{k \geq n} \sum_{j<k} a_{kj}}{\sup_{n \to \infty} \sum_{j=0}^{\infty} a_{kj}}
\]
Next, we give some examples to show that the conditions of theorem 4 are necessary.

Example 3. Let \( A \) be defined as follows:

\[
A = \begin{pmatrix}
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & & & & &
\end{pmatrix}
\]

It is easy to see that \( A \) satisfies the conditions (i) and (ii), not (iii).

Take

\[
x = (2,2,2,2,\ldots)
\]
\[
y = (2,1,1,2,1,1,2,1,1,1,2,1,1,1,2,1,1,\ldots)
\]

\( x, y \) are bounded sequences and \( x, y \in P_1 \). For \( m = 1, 2, 3, \ldots \) we have \( \mu_m(x) = \mu_m(y) = 2 \).

Hence \( \frac{\mu_m(x)}{\mu_m(y)} = 1 \). But

\[
Ax = (8,8,8,\ldots)
\]
\[
Ay = (6,3,\ldots) \quad y = (y_i) \quad y_i \leq 6 \quad i = 1, 2, \ldots
\]

This implies

\[
\frac{\mu_m Ax}{\mu_m Ay} \rightarrow \frac{8}{6} = \frac{4}{3} \neq 1, \quad \text{as} \quad n \rightarrow \infty.
\]

Example 4.

Let

\[
A = \begin{pmatrix}
1 & 1/2 & 0 & \frac{1}{4} & \frac{1}{8} & \ldots \\
0 & 1/2 & 0 & \frac{1}{4} & \frac{1}{8} & \ldots \\
\frac{1}{4} & 0 & 1/2 & 0 & \frac{1}{4} & \ldots \\
\frac{1}{8} & \frac{1}{4} & 0 & 1/2 & 0 & \ldots \\
0 & \frac{1}{8} & \frac{1}{4} & 0 & 1/2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\( A \) satisfies (i) and (ii) not (iii).

Take

\[
x = (2,2,2,\ldots)
\]
\[
y = (2,1,2,1,\ldots).
\]
$x$ and $y$ are bounded and $x, y \in P_1$. We also have

$$\frac{\mu_m x}{\mu_m y} = 1, \ m = 1, 2, \ldots$$

$$Ax = (2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$$

$$Ay = (2, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots).$$

Then, if $m$ is odd,

$$\frac{(\mu Ax)_m}{(\mu Ay)_m} = 2$$

if $m$ is even

$$\frac{(\mu Ax)_m}{(\mu Ay)_m} = 1$$

$\Rightarrow \frac{(\mu Ax)_m}{(\mu Ay)_m}$ has no limit as $m \to \infty$.

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REFERENCES


[3] Pobyvanets, I.P. Asymptotic Equivalence of Some Linear Transformation Defined by a Nonnegative Matrix and Reduced to Generalized Equivalences in the Sense of Cesáro and Abel, Mat. Fig. 28 (1980), 83-87, 123.
