FIXED POINT THEOREMS FOR SEMI-GROUPS OF
SELF MAPS OF SEMI-METRIC SPACES

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ABSTRACT. We use selected semi-groups of self maps of a semi-metric space to obtain fixed point
theorems for single maps and for families of maps – theorems which generalize results by Browder,
Jachymski, Rhoades and Walters, and others. A basic tool in our approach is the concept of commuting
maps.

KEY WORDS AND PHRASES: Semi-metric, semi-group, commuting maps, fixed points.

1. INTRODUCTION. By a semi-group of maps we shall mean a family \( H \) of self maps of a set \( X \)
which is closed with respect to composition of maps. Thus, if \( f, g \in H \), then \( f \circ g \in H \). Since
composition of maps is associative, \( H \) is indeed a semi-group with respect to composition. We shall write
\( fg \) for \( f \circ g \) and \( fx \) for \( f(x) \) when convenient and confusion is not likely.

We shall utilize the following semi-groups of maps in subsequent sections.

1.1. Let \( g : X \to X \), and \( H : \{ \, g^n : n \in \mathbb{N} \cup \{0\} \, \} \), where \( \mathbb{N} \) is the set of positive
integers, \( g^0 = i_X \) – the identity map, \( g^1 = g \) and \( g^{n+1} = g \circ g^n \).

1.2. Let \( g : X \to X \), and \( H = \{ f : X \to X \mid fg = gf \} \). \( C_g \) is a semi-group. For if
\( f, h \in C_g \), then \( (fh)g = f(hg) = f(gh) = (fg)h = g(fh) \), and thus, \( fh \in C_g \).

1.3. \( H = \{ i_X \} \), and

1.4. If \( f, g : X \to X \) and \( fg = gf \), we can let \( H = \{ f^ng^m : n, m \in \mathbb{N} \cup \{0\} \} \).

If \( H \) is a semi-group of self maps of a set \( X \) and \( a \in X \), then \( H(a) = \{ h(a) : h \in H \} \).

Consequently, if \( g : X \to X \) and \( H = O_g \), \( O_g(a) = \{ g^n(a) : n \in \mathbb{N} \cup \{0\} \} \), and is called the orbit of \( g \) at \( a \).

Just as we use semi-groups of maps to generalize the concept of orbits, we shall use semi-metric
spaces to generalize results pertaining to metric spaces. We need the following definitions.

DEFINITION 1.1 A symmetric on a set \( X \) is a function \( d : X \times X \to [0, \infty) \) such that \( d(x, y) = 0 \) iff \( x = y \) and \( d(x, y) = d(y, x) \) for all \( x, y \in X \).

Given a symmetric \( d \) on a set \( X \), we generate an induced topology \( t(d) \) for \( X \) as follows. For
\( x \in X \) and \( \epsilon > 0 \), we let \( S_\epsilon(x) = \{ y \in X : d(x, y) < \epsilon \} \). Then \( t(d) \) consists of all subsets \( U \) of \( X \) such that for each \( p \in U \), \( S_\epsilon(p) \subseteq U \) for some \( \epsilon > 0 \). Just as in the case of a metric, \( t(d) \) is a topology on \( X \).

However, if \( d \) is a symmetric, the sets \( S_\epsilon(x) \) need not be neighborhoods of \( x \). A semi-metric is a
symmetric \( d \) such that all sets \( S_\epsilon(x) \) are neighborhoods of \( x \); i.e., \( \exists U \in t(d) \) such that \( x \in U \subseteq S_\epsilon(x) \). It
is easy to verify that if \( d \) is a semi-metric, then a sequence \( \{x_n\} \) in \( X \) converges to \( x \in X \) in the topology \( t(d) \) iff \( d(x_n, x) \to 0 \). This is the property we desire. Hence, the following terminology.

**Definition 1.2** A semi-metric space is a topological space, denoted by \((X; d)\), with topology \( t(d) \) induced on the set \( X \) by a semi-metric \( d \).

For further discussion of symmetrics and semi-metrics refer to [1] or [2]. In this context, we note that a semi-metric need not be Hausdorff (or \( T_2 \)). Example 2.2 in [2] gives an instance of such a semi-metric. Since we desire uniqueness of limits, we shall in most instances require that a semi-metric space \((X; d)\) be Hausdorff. Note also that — as in metric spaces — we still say a semi-metric space \((X; d)\) is complete iff every Cauchy sequence in \( X \) converges to a point in \( X \). If \( g:X \to X \), then \((X, d)\) is \( g\)-orbitally complete iff every Cauchy sequence in \( O_g(x) \) converges to a point in \( X \) for all \( x \in X \). A function \( F:X \to [0, \infty) \) is lower semicontinuous iff \( F(x) \leq \liminf_{n \to \infty} F(x_n) \) when \( \{x_n\} \) is a sequence in \( X \) converging to \( x \).

To produce fixed points we use a contractive function \( P:[0, \infty) \to [0, \infty) \) which is nondecreasing and which satisfies: \( \lim_{n \to \infty} P^n(t) = 0 \) for each \( t \in [0, \infty) \). Throughout this paper, \( P \) will denote such a map, and \( \mathcal{P} \) will denote the family of all such maps \( P \).

2. FIXED POINT THEOREMS. The major results in this paper evolve from the following lemma.

**Lemma 2.1.** Let \( X \) be a set, \( g:X \to X \), and let \( d:X \times X \to [0, \infty) \). Let \( H \) be a semi-group of maps \( h:X \to X \) such that \( H \subseteq C_g \). Suppose that for each pair \( x, y \in X \) there is a choice of \( r = r(x, y), s = s(x, y) \in H \) for which

\[(i) \quad d(gx, gy) \leq P(d(ru, sv)), \]

then, if \( n \in \mathbb{N} \), for each pair \( x, y \in X \) \( \exists r_n, s_n \in H \) and \( u_n, v_n \in \{x, y\} \) such that

\[(ii) \quad d(g^n x, g^n y) \leq P^n(d(r_n u_n, s_n v_n)).\]

**Proof.** (ii) holds for \( n = 1 \) by (i), so suppose \( n \in \mathbb{N} \) for which (ii) is true. Then, if \( x, y \in X \),

\[
d(g^{n+1} x, g^{n+1} y) = d(g^n x, g^n y) \leq P(d(r u, s v)), \tag{2.1}
\]

where \( r, s \in H \) and \( u, v \in \{g^n x, g^n y\} \), by (i).

Specifically, \( u = g^n u_0 \) where \( u_0 \in \{x, y\} \) and \( v = g^n v_0 \) with \( v_0 \in \{x, y\} \). Thus

\[
d(r u, s v) = d((g^n u_0), (g^n v_0)) \text{ where } u_0, v_0 \in \{x, y\}. \quad \text{And since } r, s \in H \subseteq C_g,
\]

\[
d(r u, s v) = d(g^n u_0, g^n v_0) \leq P^n(d(r u_0, s v_0)), \tag{2.2}
\]

where \( r_0, s_0 \in H \) and \( u_0, v_0 \in \{r u_0, s v_0\} \).

Then \( r_n u_n \in \{(r_n r) u_0, (r_n s) v_0\} \), where \( r_n, s_n \in H \) (a semi-group). So, \( r_n u_n = r_{n+1} u_{n+1} \), where \( r_{n+1} \in H \) (i.e., \( r_{n+1} \in \{r_n r, r_n s\} \)) and \( u_{n+1} \in \{u_0, v_0\} \subseteq \{x, y\} \). Similarly, \( s_n v_n = s_{n+1} v_{n+1} \), where \( s_{n+1} \in H \) and \( v_{n+1} \in \{x, y\} \). Thus (2.2) implies

\[
d(r u, s v) \leq P^n(d(r_{n+1} u_{n+1}, s_{n+1} v_{n+1})), \text{ with } r_{n+1}, s_{n+1} \in H \text{ and } u_{n+1}, v_{n+1} \in \{x, y\}. \tag{2.3}
\]

But \( P \) is nondecreasing, therefore, (2.1) and (2.3) imply

\[
d(g^{n+1} x, g^{n+1} y) \leq P^n(d(r_{n+1} u_{n+1}, s_{n+1} v_{n+1})) = P^{n+1}(d(r_{n+1} u_{n+1}, s_{n+1} v_{n+1})), \]

with \( r_{n+1}, s_{n+1} \in H \) and \( u_{n+1}, v_{n+1} \in \{x, y\} \). Thus, (ii) is true for all \( n \) by induction. \( \square \)

**Theorem 2.1** Let \((X, d)\) be a \( T_2 \) semi-metric space. Let \( g:X \to X \) and let \((X, d)\) be \( g\)-orbitally complete. Suppose \( H \) is a semi-group of self maps of \( X \) such that \( H \subseteq C_g \), and there is an \( a \in X \) for which \( H(a) \) is bounded and \( g(H(a)) \subseteq H(a) \). If for each \( x, y \in X \) \( \exists a \) choice of \( r, s \in H \) and \( u, v \in \{x, y\} \) such that

\[(*) \quad d(gx, gy) \leq P(d(r u, s v)), \]

then...
then $g^n(a) \to c$ for some $c \in X$. If $g$ is continuous at $c$, then $g(c) = c$. If $d$ is lower semicontinuous, then $c$ is a fixed point for all $h \in H$ continuous at $c$. Moreover, if $g$ and each $h \in H$ are continuous at $c$, then $c$ is the unique common fixed point of $g$ and the family $H$.

**Proof.** We first prove that $\{g^n(a)\}$ is Cauchy. By Lemma 2.1, for each pair $n,k \in \mathbb{N}$, there is a choice of $r_n,s_n \in H$ and $u_n,v_n \in \{a, g^k(a)\}$ such that

$$d(g^n(a), g^{n+k}(a)) = d(g^n(a), g^{n+k}(a)) \leq P^n(d(r_n u_n, s_n v_n)).$$

(2.4)

Now $r_n \in H$ implies $r_n a \in H(a)$ and $r_n g^k(a) = g^k(r_n a) \in H(a)$, since $g(H(a)) \subseteq H(a)$ implies that $g^k(H(a)) \subseteq H(a)$. Thus $r_n, u_n \in H(a)$. Similarly, $s_n, v_n \in H(a)$. But $H(a)$ is bounded, so $\exists M \geq 0$ such that $d(x, y) \leq M$ for $x, y \in H(a)$. Thus, $d(r_n u_n, s_n v_n) \leq M$ for $n \in \mathbb{N}$. Then (2.4) implies

$$d(g^n(a), g^{n+k}(a)) \leq P^n(M), \quad \text{for } n, k \in \mathbb{N}$$

(2.5)

since $P$ is nondecreasing. But $P^n(M) \to 0$ as $n \to \infty$. So given $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that for any $m > n \geq n_0$, (2.5) implies $d(g^n(a), g^{m+k}(a)) \leq P^n(M) < \varepsilon$, with $m=n+k$. Consequently, $\{g^n(a)\}$ is Cauchy.

Since $(X, d)$ is $g$-orbitally complete, $g^n(a), g^{n+1}(a) \to c$ for some $c \in X$. If $g$ is continuous, $g(g^n(a)) = g^{n+1}(a) \to g(c)$; thus, $c = g(c)$ since $X$ is Hausdorff.

Now suppose that $d$ is lower semicontinuous and that $h(\in H)$ is continuous at $c$. Then, since $H \subseteq C_g$ and $g^n(a) \to c$,

$$g^n(h(a)) = h(g^n(a)) \to h(c).$$

(2.6)

But (*) and Lemma 2.1 tell us that $\exists r_n, s_n \in H$ and $u_n,v_n \in \{a, h(a)\}$ such that

$$d(g^n(a), g^n(h(a))) \leq P^n(d(r_n u_n, s_n v_n)).$$

(2.7)

Then $r_n u_n = r_n a \in H(a)$ or $r_n u_n = r_n h(a) \in H(a)$ ($r_n h \in H$, since $H$ is a semi-group). Thus, in either event, $r_n, u_n \in H(a)$. In like manner, we conclude that $s_n, v_n \in H(a)$. Then, as above, $d(r_n u_n, s_n v_n) \leq M$ for $n \in \mathbb{N}$, which implies by (2.7)

$$d(g^n(a), g^n(h(a))) \leq P^n(M) \to 0.$$

(2.8)

But since $g^n(a) \to c$, (2.6) implies that $(g^n(a), g^n(h(a))) \to (c, h(c))$ in $X \times X$. Since $d$ is lower semicontinuous, $d(c, h(c)) \leq \lim_{n \to \infty} d(g^n(a), g^n(h(a))) = 0$ by (2.8), so $h(c) = c$.

To complete the proof we have yet to show that if $c$ is a common fixed point for $g$ and every $h \in H$, then $c$ is the only such point. So suppose that $z \in X$ and that $z = g(z) = h(z)$ for all $h \in H$. Then by (*) and Lemma 3.1, we can write:

$$d(c, z) = d(g^2(c), g^2(z)) \leq P^2(d(r_n u_n, s_n v_n))$$

(2.9)

where $r_n, s_n \in H$ and $u_n, v_n \in \{c, z\}$. But then $r_n u_n \in \{c, z\}$. Similarly, $s_n, v_n \in \{c, z\}$. Therefore, $d(r_n u_n, s_n v_n) = 0$ or $d(c, z)$. Thus (2.9) says that $d(c, z) \leq P^2(d(c, z))$. Since $P^2(d(c, z)) \to 0$, $c = z$; i.e., $c$ is unique. □

The following example shows that the family $H$ in Theorem 2.1 can have fixed points other than the unique common fixed point of $g$ and $H$.

**Example 2.1.** Let $X = \{0, 1\}$, $g(x) = 0$ for $x \in X$, $h(x) = x$ for $x \in X$. Let $d(x, y) = |x - y|$ and $H = \{h^n : n \in \mathbb{N}\}$. Since $h^n(x) = x$ for $n \in \mathbb{N}$, $H = \{id\}$. Since $d(gx, gy) = 0$ for all $x, y \in X$, it is immediate that $g$ and $H$ satisfy the hypothesis of Theorem 3.1 with $a = 0$ and $1$ is a fixed point of $H$ but not of $g$.

The first corollary provides conditions necessary and sufficient to ensure that a family $H$ of continuous self maps of a semi-metric space has a fixed point.
COROLLARY 2.1. Let \((X, d)\) be a complete Hausdorff semi-metric space with \(d\) lower semicontinuous. A semi-group \(H\) of continuous self maps of \(X\) has a common fixed point iff \(H(a)\) is bounded for some \(a \in X\), and \(\exists \mathcal{P} \subseteq \mathcal{P}\) and a continuous self map \(g\) of \(X\) which satisfies the following.

1. \(H \subseteq \mathcal{C}_g\) and \(g(H(a)) \subseteq H(a)\)
2. For any \(x, y \in X\), \(\exists a\) a choice of \(r, s \in H\) and \(u, v \in \{x, y\}\) such that
   \[d(gx, gy) \leq P(d(ru, sv)).\]

**PROOF.** That the conditions are sufficient follows immediately from Theorem 2.1. To prove necessity, suppose \(a \in X\) and that \(h(a) = a\) for \(h \in H\). Then \(H(a) = \{a\}\) and is thus bounded. Let \(g(x) = a\) for \(x \in X\). It is immediate that \(gh = hg\) for all \(h \in H\), so \(H \subseteq \mathcal{C}_g\). Moreover, \(g(h(a)) = a\) for all \(h \in H\), so that \(g(H(a)) \subseteq H(a)\) and statement 1. of the Corollary holds. Statement 2. follows upon noting that \(d(gx, gy) = d(a, a) = 0\) for all \(x, y \in X\). (We can let \(P(t) = \frac{t}{2}\), e.g.)

**NOTE 2.1.** The next result and proof suggest that the function \(g\) of Theorem 2.1 may have an infinitude or unbounded set of fixed points, although \(H\) may have just one. Example 3.1 in the next section confirms this.

**COROLLARY 2.2.** Let \((X, d)\) be a complete Hausdorff semi-metric space with \(d\) lower semicontinuous. A semi-group \(H\) of continuous self maps of \(X\) has a common fixed point provided \(a \in X\) such that \(H(a)\) is bounded, and for any \(x, y \in X\), \(r, s \in H\) and \(u, v \in \{x, y\}\) such that

\[d(x, y) \leq P(d(ru, sv)).\]

**PROOF.** Let \(g = id\), the identity map.

**COROLLARY 2.3.** Let \(g\) be a self map of a metric space \((X, d)\) which is \(g\)-orbitally complete. If \(\exists a \in X\) such that \(O_g(a)\) is bounded and \(k \in N\) such that for each pair \(x, y \in X\) there is a choice of \(n = n(x, y), m = m(x, y) \in N\) and \(u, v \in \{x, y\}\), for which

\(^(*)\) \[d(g^n x, g^n y) \leq P(d(g^m u, g^m v))\]

then \(g^n(a) \to c\) for some \(c \in X\). Moreover, if \(x_o \in X\) and \(O_g(x_o)\) is bounded, then \(g^n(x_o) \to c\). If \(g\) is continuous at \(c\), \(c\) is the unique fixed point of \(g\).

**PROOF.** Let \(H = O_g\). Note that \(H \subseteq C^g\), \(g(H(a)) = g(O_g(a)) \subseteq O_g(a)\), and that \(g^k(H(a)) \subseteq H(a)\). Since \(d\) is a metric, \(d\) is actually uniformly continuous [3]. Thus, \(g^m(a) \to c\) as \(m \to \infty\) for some \(c \in X\), by Theorem 2.1 applied to \(g^k\). If \(g\) is continuous at \(c\), each \(g^n \in H(= O_g)\) is continuous at \(c\). So, as an element of \(H\), \(g(c) = c\) by Theorem 2.1..

We have yet to prove that \(g^n(a) \to c\) and that \(g^n(x_o) \to c\) for \(x_o\) with \(O_g(x_o)\) bounded. To see that \(g^n(a) \to c\), let \(\varepsilon > 0\). Since \((g^n)^m(a) \to c\) as \(m \to \infty\), \(\exists m_1 \in N\) such that

\[d(g^n(a), c) < \varepsilon/2, \text{ for } m > m_1\]

(2.10)

By Lemma 2.1 and \(^(*)\) of the hypothesis, if \(m \in N\), for each pair \(x, y \in X\) there exist \(r_m, s_m \in H\) and \(u_m, v_m \in \{x, y\}\) such that

\[d((g^n)^m(x), (g^n)^m(y)) \leq P^m(d(r_m u_m, s_m v_m)).\]

(2.11)

Since \(O_g(a)\) is bounded, \(\exists M \geq 0\) such that

\[d(r_m u_m, s_m v_m) \leq M \text{ if } r_m u_m \text{ and } s_m v_m \text{ are in } O_g(a)\]

(2.12)

Now \(P^m(M) \to 0\) as \(m \to \infty\), so we can choose \(m_0 \in N\) such that

\[m_0 > m_1 \text{ and } P^{m_0}(M) < \varepsilon/2.\]

(2.13)

Let \(n > km_0\). Then \(n = km_0 + t_n\) for some \(t_n \in N\), and (2.11) implies

\[d(g^n(a), g^{km_0}(a)) = d(g^{km_0}(g^n(a)), g^{km_0}(a)) \leq P^{km_0}(d(r_m u_m, s_m v_m))\]

(2.14)
where \( r_m \in H = O_g \), \( u_m \in \{a, g^k(a)\} \), so that \( r_m u_m \in O_g(a) \). Similarly, \( s_m v_m \in O_g(a) \).

Therefore, (2.12) implies that

\[
d(r_m u_m, s_m v_m) \leq M,
\]

and since \( P \) is nondecreasing, we have

\[
P^m( d(r_m u_m, s_m v_m) ) \leq P^m(M) < e/2,
\]

by (2.13). Thus (2.14) implies

\[
d(g^m(a), g^{km}(a)) < e/2.
\]

But then (2.10) with \( m = m_0 \) and the triangle inequality imply that

\[
d(g^n(a), g^m(a)) < e, \quad \text{since } m_0 > m_1.
\]

We therefore conclude that \( g^a(a) \to c \)

If \( x_0 \in X \) such that \( O_g(x_0) \) is bounded, the above argument shows us that \( \exists \ p \in X \) such that

\( g^a(x_0) \to p \). To see that \( p = c \), first observe that \( S = O_g(x_0) \cup O_g(a) \) is also bounded since \( d \) is a metric; i.e., \( \exists \ M_0 \geq 0 \) such that \( d(x, y) \leq M_0 \) if \( x, y \in X \). We can therefore apply (**) and Lemma 2.1 as before to conclude that

\[
d(c, p) = \lim_{m \to \infty} d(g^m(a), g^m(x_0)) \leq \lim_{m \to \infty} P^m(M_0) = 0.
\]

The following example shows that the hypothesis in Corollary 2.3 that the orbit \( O_g(a) \) be bounded for at least one \( a \in X \) is indeed necessary.

**Example 2.2.** Let \( X = [1, \infty) \), \( P(t) = t^2/2 \) for \( t \in [0, \infty) \), \( d(x, y) = |x - y| \) and \( g(x) = 3x \) for \( x, y \in X \). Then \( P^2(t) = t^2/2 \to 0 \) as \( n \to \infty \), \( g(x) \to X \) and \( d(g(x), g(y)) = |gx - gy| = 3|x - y| \)

\[
\leq 9/2 \quad \text{for } x, y \in X.
\]

But since \( g^a(x) = 3^n x \to \infty \) for each \( x \in X \), \( O_g(x) \) is bounded for no \( x \in X \) and \( g \) has no fixed point.

In [4] Rhoades and Watson introduced the concept of a "generalized contraction".

**Definition 2.1.** Let \( (X, d) \) be a metric space. A function \( f : X \to X \) is a generalized contraction (with respect to \( Q \)) if \( \exists \ p, q \in N \) such that for all \( x, y \in X \)

(i)

\[
d(f^p x, f^q y) \leq Q(M(x, y)),
\]

where

\[
M(x, y) = \max\{ d(f^i x, f^j y), d(f^i x, f^j x), d(f^j y, f^j y) : 0 \leq i, j \leq p, 0 \leq i, j \leq q \}.
\]

(\( Q \) is a nondecreasing function \( Q : [0, \infty) \to [0, \infty) \) such that \( Q(s) < s \) for \( s > 0 \).)

**Note 2.2.** Jachymski [5] studied the relation (i) and observed that it satisfies

(ii)

\[
d(f^r x, f^r y) \leq Q(\max\{ d(f^i u, f^j v) : 0 \leq i, j \leq r \text{ and } u, v \in \{x, y\}\})
\]

where \( r = \max\{p, q\} \), since \( Q \) is nondecreasing. But (ii), and hence (i), satisfy the relation (**) in Corollary 2.3. In fact, the following theorem by Jachymski — except the last sentence therein — is a consequence of Corollary 2.3. This last sentence refers to (9) which is essentially (i) above with the restriction that either \( i, i' \in \{0, p\} \) or \( j, j' \in \{0, q\} \).

**Theorem 4.** ([5]) Let \( f \) be a generalized contraction and let \( (X, d) \) be \( f \)-orbitally complete. If

\[
\lim_{n \to \infty} Q^{n}(s) = 0 \quad \text{for } s \in [0, \infty) \text{ and there exists a point } x_0 \in X \text{ with a bounded orbit, then the sequence } \{f^n x_0\} \to z \quad \text{in } X.
\]

Moreover, for any \( x \in X \) with a bounded orbit, \( f^n x \to z \). Furthermore, if \( f \) satisfies (9) \( z \) is the unique fixed point of \( f \).

The following example shows that if we use the more general contractive property (**) of Corollary 2.3, continuity at \( c \) or restrictions of the ilk found in (9) of Theorem 4 are needed to ensure that the point \( c \) (or \( z \)) is a fixed point.

**Example 2.3.** Let \( X = [0, 1] \) and let \( d(x, y) = |x - y| \). Define \( g : X \to X \) by \( g(x) = \frac{1}{2}(x + 1) \) for \( x \in [0, 1) \) and \( g(1) = \frac{1}{2} \). Then it is easy to see that \( g^n(x) = (x - 1)2^{-n+1} \) (\( x \neq 1 \)), and \( g^n(1) = 1 - 2^{-n} \) for \( n \in N \). Thus, \( g^n(a) \to 1 \) for any \( a \in X \). Since \( X \) is bounded, \( O_g(a) \) is bounded for each \( a \in X \). Thus, to see that the hypothesis of Corollary 2.3 is satisfied, we need only to verify that (**) holds. A check shows that
\[ d(g^2x, g^2y) = \frac{1}{2} d(gx, gy) \quad \text{for all } x, y \in X. \] (2.15)

Thus, (**) holds trivially with \( k=2 \), \( n = m = 1 \), and \( u = x, v = y \) for all \( x, y \in [0, 1] \). However, \( g^n(a) \to 1 \) for any \( a \), but \( g \) is not continuous at 1 and 1 is not a fixed point of \( g \). Note that property (9) of Jachymski's Theorem 4. does not hold since in this instance \( p = q = 2 \), and \( g^2, g^0 \) do not appear in the right member of (2.15).

3. THE BOUNDED CASE. The following is an example of a function \( g \) and a family \( H \) which satisfy the hypothesis of Theorem 2.1 and for which the set \( F_g \) — the set of fixed points of \( g \) — is not bounded. We then consider the significance of this phenomenon.

**Example 3.1.** Let \( X = [0, \infty) \) and \( d(x, y) = |x - y| \) for \( x, y \in X \). Let \( g(x) = x \) for \( x \in X \); i.e., \( g \) is the identity map. So \( F_g = [0, \infty) \), and is unbounded. Let \( P(t) = t/2 \) for \( t \in [0, \infty) \) and define \( h_n(x) = nx \) for \( x \in X \) and \( n \in \mathbb{N} \). If \( H = \{ h_n : n \in \mathbb{N} \} \), and \( h_n, h_m \in H \), then \( h_n h_m(x) = h_n(h_m(x)) = h_{nm}(x) \). Thus \( h_{nm} = h_{nm}(x) \), so that \( H \) is indeed a semi-group. Since \( g \) is the identity, the conditions \( H \subseteq C_g \) and \( g(H(a)) \subseteq H(a) \) (for any \( a \in X \)) are satisfied trivially. Moreover, \( d(gx, gy) = |x - y| \leq \frac{1}{2} |x - y| = \frac{1}{2} |3x - 3y| = P(d(h_3x, h_3y)) \), so that (**) and hence the hypothesis of Theorem 2.1 is satisfied. \( a=0 \) is the unique fixed point for \( g \) and \( H \), but \( g \) has an infinitude of other fixed points.

In the remainder of the paper, if \((X, d)\) is a \( T_2 \) semi-metric space and \( g : X \to X \), we shall say that

\( g \) has property \( P \) relative to a semi-group \( H \) of self maps of \( X \) iff for each pair \( x, y \in X \) exists \( r, s \in H \) and \( u, v \in \{ x, y \} \) such that

\[ (*) \quad d(gx, gy) \leq P(d(ru, sv)). \]

**Note 3.1.** If a function \( g : X \to X \) has property \( P \) relative to a semi-group \( H \) of self maps of \( X \) for which \( H \subseteq C_g \), Lemma 2. implies that if \( n \in \mathbb{N} \), for any pair \( x, y \in X \) there exist \( r_n, s_n \in H \) and \( u_n, v_n \in \{ x, y \} \) such that

\[ (***) \quad d(g^n x, g^n y) \leq P^n(d(r_n u_n, s_n v_n)). \]

**Proposition 3.1.** Let \((X, d)\) be a \( T_2 \) semi-metric space and let \( g : X \to X \). Suppose \( H \) is a semigroup of self maps of \( X \) such that \( F_g \) is nonempty and bounded, then

(i) \( F_g \) is a singleton \( \{ c \} \), and (ii) \( c = g(c) = h(c) \) for all \( h \in H \).

**Proof.** To prove (i), we first note that \( h(F_g) \subseteq F_g \) for all \( h \in H \). For if \( h \in H \) and \( a = g(a) \), then \( h(a) = h(g(a)) = h(a) \), so that \( a \in F_g \). Moreover, since \( F_g \) is bounded, \( \exists M \geq 0 \) such that \( d(a, c) \leq M \) for \( a, c \in F_g \).

Now by hypothesis, \( \exists c \in F_g \). We assert that c is unique. For suppose \( a \in F_g \). Then Note 3.1 says that we can choose \( u_n v_n \in \{ a, c \} \) and \( r_n, s_n \in H \) such that

\[ d(a, c) = d(g^n(a), g^n(c)) \leq P^n(d(r_n u_n, s_n v_n)). \] (3.1)

But since \( h(F_g) \subseteq F_g \) for \( h \in H \), and since \( a, c \in F_g \), \( r_n u_n, s_n v_n \in F_g \). So by the above, \( d(r_n u_n, s_n v_n) \leq M \).

Therefore, since \( P \) and hence \( P^n \) is nondecreasing, (3.1) yields:

\[ d(a, c) \leq P^n(M) \quad \text{for } n \in \mathbb{N}. \] (3.2)

Since \( P^n(M) \to 0 \) as \( n \to \infty \), (3.2) implies that \( a = c \).

(ii) is an immediate consequence of (i), since \( (h \in H) \Rightarrow h(F_g) \subseteq F_g \). Therefore, if \( h \in H \), \( h(c) \in \{ c \} \); i.e., \( h(c) = c \).
COROLLARY 3.1. Let \((X, d)\) be a bounded and complete \(T_2\) semi-metric space. Let \(g: X \to X\) and let \(H\) be a semi-group of self-maps of \(X\) such that \(H \subset C_g\) and \(g(H(a)) \subset H(a)\) for \(a \in X\). If \(g\) has property \(P\) relative to \(H\), then for each \(x \in X\) there exists a point \(c_x \in X\) such that \(g^n(x) \to c_x\). If \(g\) is continuous at one such \(c_x\), then \(F_g\) is a singleton, \(\{c\}\), and \(c_x = c\) for all \(x\). Moreover, \(c = h(c)\) for all \(h \in H\).

PROOF. Since \(X\) is bounded, \(H(x)\) is bounded for each \(x \in X\). Therefore, Theorem 2.1 implies that \(g^n(x) \to c_x\) for some \(c_x \in X\). If \(g\) is continuous at one such \(c_x\), then \(g(c_x) = c_x\). But then \(F_g\) is bounded and nonempty, so that Proposition 3.1 implies that \(F_g = \{c\}\), a singleton, and that \(c = h(c)\) for all \(h \in H\).

Corollary 3.1 has Theorem 1.[6] by Browder and a result by Zitarosa [7] on contractive self-maps of a bounded complete metric space as special cases with \(H = \{id\}\).

The proof of our next theorem, as did the proof of Corollary 2.3, requires that the union of two bounded sets be bounded. So we again need a metric. Also, observe that in Example 3.1 the set \(H(a) = \{na: n \in \mathbb{N}\}\) is unbounded for \(a \neq 0\).

THEOREM 3.1. Let \(g\) be a self map of a metric space \((X, d)\) which is \(g\)-orbitally complete. Suppose that \(H\) is a semi-group of self-maps of \(X\) such that \(H \subset C_g\) and that \(g\) has property \(P\) relative to \(H\). If \(g(H(a)) \subset H(a)\) and \(H(a)\) is bounded for all \(a \in X\), then \(g\) has a contractive point \(c\), i.e., \(g^n(x) \to c\) for all \(x \in X\). Moreover, \(c = g(c) = h(c)\) for all \(h \in H\) if \(g\) is continuous at \(c\).

PROOF. Let \(a \in X\). By Theorem 2.1, since \(H(a)\) is bounded, \(g^n(a) \to c\) for some \(c \in X\). But \(g^n(x) \to c_x \in X\) for any \(x \in X\) since \(H(x)\) is bounded. We show \(c_x = c\) for any \(x \in X\). To this end, let \(x \in X\). Then \(H(x) \cup H(a)\) is bounded. Since \(g\) has property \(P\) relative to \(H\) and \(H \subset C_g\), Note 3.1 implies that for all \(n \in \mathbb{N}\) we have:

\[
d(g^n(a), g^n(x)) \leq P^n(d(r_nu_n, s_nv_n)),
\]

where \(r_n, s_n \in H\) and \(u_n, v_n \in \{a, x\}\); hence, \(r_nu_n, s_nv_n \in H(a) \cup H(x)\) for \(n \in \mathbb{N}\). But \(H(a) \cup H(x)\) is bounded, and so \(\exists M \geq 0\) such that \(d(r_nu_n, s_nv_n) \leq M\) for all \(n\). Thus, \(P^n(d(r_nu_n, s_nv_n)) \leq P^n(M) \to 0\) as \(n \to \infty\). Hence (3.3) and the above imply:

\[
d(c, c_x) = \lim_{n \to \infty} d(g^n(a), g^n(x)) = 0.
\]

Thus \(c = c_x\). If \(g\) is continuous at \(c\), then \(c = g(c)\). But since \(g^n(x) \to c\) for all \(x\), \(c\) is the only fixed point of \(g\). Therefore, \(c = h(c)\) for all \(h \in H\) by Proposition 3.1.

COROLLARY 3.2. Let \(g\) be a self map of a metric space \((X, d)\) which is \(g\)-orbitally complete. Suppose that \(O_g(x)\) is bounded for all \(x \in X\). If \(g\) has property \(P\) relative to \(O_g\), then \(\exists z \in X\) such that \(g^n(x) \to z\) for all \(x \in X\). \(z\) is a unique fixed point of \(g\) iff the function \(F(x) = d(x, g(x))\) is lower semicontinuous at \(z\).

PROOF. Since trivially, \(O_g \subset C_g\) and \(g(O_g(x)) \subset O_g(x)\) for all \(x \in X\), Corollary 3.2 follows immediately from Theorem 3.1 (with the observation that the last statement is a well known consequence of \("g^n(x) \to z\"\)).

We conclude with a theorem (rephrased) by Jachymski [5] which generalizes theorems of Rhoades and Watson [4], and which is a consequence of our Corollary 2.3.

THEOREM 2. [5] Assume that \(f\) is a generalized contraction, and \((X, d)\) is \(f\)-orbitally complete. If \(\lim_{n \to \infty} Q^n(s) = 0\) for \(s \in [0, \infty)\) and \(\lim_{s \to \infty} (s - Q(s)) = \infty\), then there exists \(z \in X\) such that \(f^n(x) \to z\) for any \(x \in X\). \(z\) is a unique fixed point of \(f\) if and only if the function \(F(x) = d(x, f(x))\) is lower semicontinuous at \(z\).

To see that Theorem 2. [5] does indeed follow from Corollary 2.3, first observe that (as noted before) a generalized contraction satisfies (\(\ast\)) of Corollary 2.3. Moreover, Lemma 3. [5] tells us that if...
\[ \lim_{s \to \infty} (s - Q(s)) = \infty, \] then the orbits \( O_f(x) \) are bounded for all \( x \in X \). Therefore, Corollary 2.3 assures us that \( \exists z \in X \) such that \( f^n(x) \to z \) for all \( x \in X \). The assertion that \( z \) is the unique fixed point of \( f \) follows as in the proof of Corollary 3.2.

4. RETROSPECT. In closing we emphasize the general nature and utility of the semi-groups \( H \) of self maps employed. For example, in Corollary 2.2 \( H \) is any family of continuous self maps closed under composition with \( H(a) \) bounded at some one point \( a \in X \) – no commutativity requirements are imposed. Corollary 2.3 demonstrates the utility of options provided by \( H \) in letting \( H = O_a \). And Example 1.4 indicates how, when given a map \( g : X \to X \), we can generate semigroups \( H \) which satisfy \( g(H(a)) \subseteq H(a) \).

Note also that Hausdorff semi-metric spaces \((X, d)\) generalize metric spaces, even if the semi-metric \( d \) is lower semi-continuous. In fact, Cook [8] provides an example of a semi-metric space with a continuous semi-metric which is developable but not normal, and hence not a metric.

A final comment. The semi-group \( C_g \) has been used to some extent in fixed point research. See, e.g., [9, 10, 11].

REFERENCES


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