SYMMETRIC AND PERMUTATIONAL GENERATING SET OF THE GROUPS A_{kn+1} AND S_{kn+1} USING S_n AND AN ELEMENT OF ORDER k

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ABSTRACT. In this paper we will show how to generate A_{kn+1} and S_{kn+1} using a copy of S_n and an element of order k in A_{kn+1} and S_{kn+1} respectively, for all positive integers $n \geq 2$ and all positive integers $k \geq 2$. We will also show how to generate A_{kn+1} and S_{kn+1} symmetrically using n elements each of order k , for all $n \geq 2$ and all even integers $k \geq 2$.

KEY WORDS AND PHRASES: Symmetric generators, Group presentation, Doubly transitive groups.

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1. INTRODUCTION

Hammas [1] showed that A_{2n+1} can be presented as

$$G = A_{2n+1} = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^2 = [T, S_{n-1}] = 1 \rangle$$

for $n = 4, 6$, where $[T, S_{n-1}]$ means that T commutes with Y and with $X^2 Y X$, (the generators of S_{n-1}). The relations of the symmetric group $S_n = \langle X, Y \rangle$ of degree n are found in Coxeter and Moser [2]. Some relations must be added to the presentation that generates A_{2n+1} in order to complete the coset enumeration. Also Hammas [1] showed that, for $n = 4, 6$, the group A_{2n+1} can be symmetrically generated by n elements each of order 2 and of the form $T_0, T_1, \dots, T_{n-1}$, where $T_i = T^{X^i} = X^{-i} T X^i$ and T, X satisfy the relations of the group A_{2n+1} . The set $\{T_0, T_1, \dots, T_{n-1}\}$ is called the symmetric generating set of A_{2n+1} (see the Definition 2.1 in Section 2).

Hammas [3] showed that A_{2n+1} can be presented as

$$A_{2n+1} = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^2 = [T, Y] = [T, X^2 Y X] = (XT)^{2n+1} = (YT_{n-2})^{10} \rangle$$

when n is an even integer and S_{2n+1} can be presented as

$$S_{2n+1} = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^2 = [T, Y] = [T, X^2 Y X] = (XT)^{n(n+1)} = (YT_{n-2})^{10} \rangle.$$

when n is odd. Note that the order of the third generator, T , was always 2.

Also, it has been shown by Hammas [3] that for all $n \geq 2$ the groups A_{2n+1} and S_{2n+1} can be symmetrically generated using n elements each of order 2, and of the form $T_0, T_1, \dots, T_{n-1}$, where $T_i = T^{X^i} = X^{-i} T X^i$ and T, X satisfy the relations of the groups A_{2n+1} and S_{2n+1} .

In this paper, we give a generalization of the results obtained by Hammas [1-3]. We will show that, for all $k \geq 2$ and for all $n \geq 2$, the group generated by X, Y and T is the alternating group A_{kn+1} when n and k are all even integers and is the symmetric group S_{kn+1} otherwise. Moreover, relations will be given to show that, for all $k \geq 2$ and for all $n \geq 2$, the group

$$G = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^k = [T, S_{n-1}] = 1 \rangle$$

is A_{kn+1} when n and k are both even and S_{kn+1} otherwise. We give permutations that generate A_{kn+1} and S_{kn+1} which satisfy the conditions given in the presentation of the group G . Further, we prove that, when k is an even integer, G can be symmetrically generated by n permutations each of order k of the form $T_0, T_1, \dots, T_{n-1}$, where $T_i = T^{X^i} = X^{-i} T X^i$, satisfying the condition that T_0 commutes with the generators of the group S_{n-1} .

2. PRELIMINARY RESULTS

THEOREM 2.1. Let $1 \leq a \neq b \leq n$ be any integers. Let G be the group generated by the n -cycle $(1, 2, \dots, n)$ and the 3-cycle (n, a, b) where the highest common factor $\text{hcf}(n, a, b) = 1$. If n is an odd integer then $G = A_n$ while, if n is even, then $G = S_n$.

DEFINITION 2.1. Let G be a group and $\Gamma = \{T_0, T_1, \dots, T_{n-1}\}$ be a subset of G where $T_i = T^{X^i} = X^{-i} T X^i$ for all $i = 0, 1, \dots, n-1$. Let S_n - a copy of the symmetric group of degree n - be the normalizer in G of the set Γ . We define Γ to be a symmetric generating set of G if and only if $G = \langle \Gamma \rangle$ and S_n permutes Γ doubly transitively by conjugation, i.e., Γ is realizable as an inner automorphism.

3. PERMUTATIONAL GENERATING SET OF A_{kn+1} AND S_{kn+1}

THEOREM 3.1. For all $n \geq 2$ and all $k \geq 2$, A_{kn+1} can be generated using a copy of S_n and an element of order k in A_{kn+1} when n and k are both even and S_{kn+1} can be generated using a copy of S_n and an element of order k in S_{kn+1} if n or k is odd.

PROOF. Let $X = (1, 2, \dots, n)(n+1, n+2, \dots, 2n) \dots ((k-1)n+1, (k-1)n+2, \dots, kn)$, $Y = (n-1, n) \dots (kn-1, kn)$ and $T = (1, n+1, 2n+1, 3n+1, \dots, (k-2)n+1, kn+1)(2, n+2, 2n+2, \dots, (k-1)n+2) \dots (n, 2n, \dots, kn)$ be three permutations; the first of order n , the second of order 2 and the third of order k . Let H be the group generated by X and Y . By a result of Burnside and Moore, (see Coxeter and Moser[2]), the group H is the symmetric group S_n . Let G be the group generated by X, Y and T . We have two cases:

Case 1 Let k be an odd integer. Let $\alpha = [X, T]$. Then $\alpha = (1, (k-1)n+1, kn+1, (k-1)n+2, 2)$. Let $\beta = \alpha^3 \alpha^T$. Then

$$\beta = (1, (k-1)n+2)(2, kn+1)(n+1, (k-1)n+1, n+2).$$

Let $\delta = \alpha X \beta^T \beta^{T^2} \dots \beta^{T^{k-3}} \alpha Y^X$. Hence

$$\begin{aligned} \delta = & (1, kn, 3n, n+2, \dots, 2n-1, n+1, 2n+2, \dots, 3n-1, 2n+1, 2n, 5n, 3n+2, \dots, 4n-1, 3n+1, 4n+2, \dots, \\ & 5n-1, 4n+1, 4n, 7n, 5n+2, \dots, 6n-1, 5n+1, 6n+2, \dots, 7n-1, 6n+1, 6n, 9n, \dots, (k-6)n+2, \dots, \\ & (k-5)n-1, (k-6)n+1, (k-5)n+2, \dots, (k-4)n-1, (k-5)n+1, (k-5)n, (k-2)n, (k-4)n+2, \dots, \\ & (k-3)n-1, (k-4)n+1, (k-3)n+2, \dots, (k-2)n-1, (k-3)n+1, (k-3)n, (k-2)n+2, \dots, \\ & (k-1)n-1, (k-2)n+1, (k-1)n, kn+1, (k-1)n+3, \dots, \\ & kn-1, (k-1)n+1, n, 2, (k-1)n+2, 3, \dots, n-1) \end{aligned}$$

which is a cycle of length $kn+1$. Let $K = \langle \delta, \beta^2 \rangle$. We claim that K is either A_{kn+1} or S_{kn+1} . To show this, let θ be the mapping which takes the element in the position i of the permutation β into the element i in the permutation $(1, 2, \dots, kn+1)$. Under the mapping θ , the group K will be mapped into the group

$$\theta(K) = \langle (1, 2, \dots, kn+1), (n+1, 4, (k-1)n+1) \rangle.$$

Since k is an odd integer the highest common factor $\text{hcf}(n+1, 4, (k-1)n+1) = 1$. Hence by Theorem 2.1, if n is an odd integer then $\theta(K)$ is S_{kn+1} . Hence G is S_{kn+1} . But if n is an even integer then $\theta(K)$ is A_{kn+1} . Since k is an odd integer, Y is an odd permutation. The action of the generators of A_{kn+1} on Y is not trivial and therefore G is the symmetric group S_{kn+1} .

Case 2 Let k be an even integer. Let $\alpha = [X, T]$. Then $\alpha = (1, (k-1)n+1, kn+1, (k-1)n+2, 2)$. Let $\beta = \alpha^3 \alpha^T$. Then

$$\begin{aligned} \beta &= (1, (k-1)n+2)(2, kn+1)(n+1, (k-1)n+1, n+2). \\ \text{Let } \delta &= \alpha X \beta^T \beta^{T^3} \dots \beta^{T^{(k-4)/2}} \alpha Y^X. \text{ Hence} \\ \delta &= (1, 2, 2n, n, n+2, \dots, 2n-1, n+1, kn, 4n, 2n+2, \dots, 3n-1, 2n+1, 3n+2, \dots, \\ &4n-1, 3n+1, 3n, 6n, 4n+2, \dots, 5n-1, 4n+1, 5n+2, \dots, 6n-1, 5n+1, 5n, 8n, \dots, \\ &(k-6)n+2, \dots, (k-5)n-1, (k-6)n+1, (k-5)n+2, \dots, (k-4)n-1, (k-5)n+1, (k-5)n, \\ &(k-2)n, (k-4)n+2, \dots, (k-3)n-1, (k-4)n+1, (k-3)n+2, \dots, (k-2)n-1, (k-3)n+1, \\ &(k-3)n, (k-2)n+2, \dots, (k-1)n-1, (k-2)n+1, (k-1)n, kn+1, (k-1)n+3, \dots, \\ &kn-1, (k-1)n+1, (k-1)n+2, 3, \dots, n-1) \end{aligned}$$

which is a cycle of length $kn+1$. Let $K = \langle \delta, \beta^2 \rangle$. Using the same method used above we can easily show that K is the alternating group A_{kn+1} . Now, since k is an even integer, then, if n is an even integer too, G has to be the alternating group A_{kn+1} or a proper subgroup of it. Since K is the alternating group A_{kn+1} then G is the alternating group A_{kn+1} . But if n is an odd integer then T , the third generator of G , is an odd permutation. Since the action of the generators of the group K on the element T is not trivial, the group $\langle \delta, \beta^2, T \rangle$ is the symmetric group S_{kn+1} . Hence G is the symmetric group S_{kn+1} .

4. SYMMETRIC PERMUTATIONAL GENERATING SET OF A_{kn+1} and S_{kn+1}

THEOREM 4.1. Let X, Y and T be the permutations described in Theorem 3.1 where $T^k = 1$. Let $\Gamma = \{T_0, T_1, \dots, T_{n-1}\}$, where $T_i = T^{X^i}$. Let k be an even integer. If n is an even integer too, then the set Γ generates the alternating group A_{kn+1} symmetrically, while, if n is an odd integer, then the set Γ generates the symmetric group S_{kn+1} symmetrically.

PROOF. Let $T_0 = (1, n+1, 2n+1, 3n+1, \dots, kn+1)(2, n+2, 2n+2, \dots, (k-1)n+2) \dots (n, 2n, \dots, kn)$, $T_1 = T^X = (1, n+1, 2n+1, \dots, (k-1)n+1)(2, n+2, 2n+2, \dots, kn+1) \dots (n, 2n, \dots, kn)$, $\dots$, $T_{n-1} = T^{X^{n-1}} = (1, n+1, 2n+1, \dots, (k-1)n+1)(2, n+2, 2n+2, \dots, (k-1)n+2) \dots (n, 2n, \dots, kn+1)$. Let $H = \langle \Gamma \rangle$. We claim that $H \cong A_{kn+1}$ or S_{kn+1} . To show this, suppose first that n is an odd integer. Let $\ell = \frac{k}{2}$ if $\frac{k}{2}$ is even and $\ell = \frac{k}{2} + 1$ if $\frac{k}{2}$ is odd and let $r = \ell - 1$ if $\frac{k}{2}$ is odd and $r = \ell + 1$ if $\frac{k}{2}$ is even. Consider the element $\alpha = (T_1 T_2 \dots T_{n-2})_{T_{n-1}}$. We find that

$$\begin{aligned} \alpha &= (1, 2n+1, 4n+1, \dots, \ell n+1, (\ell+1)n+1, (\ell+3)n+1, \dots, (k-1)n+1, n+1, 3n+1, 5n+1, \dots, \\ &n+1, (r+1)n+2, (r+3)n+2, \dots, (k-1)n+2, n+2, 3n+2, 5n+2, \dots, rn+2, (r+3)n+2, (r+5)n+2, \dots, \end{aligned}$$

$(k-2)n+2, 2, 2n+2, 4n+2, \dots, (\ell-2)n+2, \ell n+3, (\ell+2)n+3, \dots, (k-2)n+3, 3, 2n+3, 4n+3, \dots,$
 $(\ell-2)n+3, (\ell+1)n+3, (\ell+3)n+3, \dots, (k-1)n+3, n+3, 3n+3, 5n+3, \dots, (r-2)n+3, rn+4, (r+2)n+4, \dots,$
 $(k-1)n+4, n+4, 3n+4, 5n+4, \dots, (r-2)n+4, (r+1)n+4, (r+3)n+4, \dots, (k-2)n+4, 4, 2n+4, 4n+4, \dots,$
 $(\ell-4)n+4, (\ell-2)n+5, \ell n+5, (\ell+2)n+5, \dots, (k-2)n+5, 5, 2n+5, 4n+5, \dots, (\ell-4)n+5, (\ell-1)n+5,$
 $(\ell+1)n+5, (\ell+3)n+5, \dots, (k-1)n+5, n+5, 3n+5, 5n+5, \dots, (r-4)n+5, (r-2)n+6, rn+6, (r+2)n+6, \dots,$
 $(k-1)n+6, n+6, 5n+6, 7n+6, \dots, (r-4)n+6, (r-1)n+6, (r+1)n+6, (r+3)n+6, \dots, (k-2)n+6, 6, 2n+6,$
 $4n+6, \dots, (\ell-6)n+6, (\ell-4)n+7, (\ell-2)n+7, \ell n+7, (\ell+2)n+7, \dots, (k-2)n+7, 7, 2n+7, 4n+7, \dots,$
 $(\ell-6)n+7, (\ell-3)n+7, (\ell-1)n+7, (\ell+1)n+7, \dots, (k-1)n+7, n+7, 3n+7, 5n+7, \dots, (r-6)n+7,$
 $(r-4)n+8, (r-2)n+8, rn+8, \dots, (k-1)n+8, n+8, 5n+8, 7n+8, \dots, (r-6)n+8, (r-3)n+8, (r-1)n+8,$
 $(r+1)n+8, (r+3)n+8, \dots, (k-2)n+8, \dots, n-2, \dots, n, 3n, \dots, (k-1)n, kn, 2n, 4n, \dots, (k-2)n, kn+1, n-1,$
 $2n-2, 3n-3, \dots, (n-2)n-(n-2), (n-1)n+1, (n+1)n+1, (n+3)n+1, \dots, (k-2)n+1$

which is a cycle of length $kn+1$. Let $\beta = T^{-1}T^X$. Therefore $\beta = (1, 2, (k-1)n+2, kn+1, (k-1)n+1)$ which is a cycle of length 5. Let $\gamma = \beta^3\beta^T$. Since $\gamma = (2, n+2, n+1)$ then using the same method used in Theorem 3.1' above we get $H_1 = \langle \alpha, \gamma, T \rangle \cong S_{kn+1}$. Hence $H \cong H_1 \cong \Theta(H_1) \cong S_{kn+1}$. In the same way we can show that, when n is an even integer, $H \cong A_{kn+1}$.

The above results can be summarised in the following table:

	n	k	$G \cong \langle X, Y, T \rangle$	$\langle X, T \rangle$	$\langle \Gamma \rangle$
1	even	even	A_{kn+1}	A_{kn+1}	A_{kn+1}
2	even	odd	S_{kn+1}	S_{kn+1}	A_{kn+1}
3	odd	even	S_{kn+1}	S_{kn+1}	S_{kn+1}
4	odd	odd	S_{kn+1}	A_{kn+1}	A_{kn+1}

where

$$\begin{aligned}
 G = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^k = [T, Y] = [T, X^2 Y X] = ([X, T] X)^n = (Y X)^{n-1}, [X, T]^5 = \\
 (T^X T^{-1})^5 = (T T^{-1} [X, T]^2 [X, T]^T)_{T^{-1}}^k = (T^{([X, T]^2 [X, T]^T)_{T^{-1}}})^6 = (T [X, T]^X)^r = 1 >
 \end{aligned}$$

where $r = k(k-1)$ when k is odd and $r = 2k(k+1)$ when k is even for all $n, k \geq 3$.

From the above table we can see that in the case when k is an odd integer the set $\Gamma = \{T_0, T_1, \dots, T_{n-1}\}$ cannot generate the symmetric group S_{kn+1} symmetrically. As a matter of fact, as we verified using the GAP package, the set Γ generates the alternating group A_{kn+1} symmetrically. But unfortunately we haven't found a hand proof of this case yet.

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