COMPLETE CONVERGENCE FOR B -VALUED L^p -MIXINGALE SEQUENCES

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ABSTRACT. Under weaker conditions of probability, we discuss in this paper the complete convergence for the partial sums and the randomly indexed partial sums of B -valued L^p -mixingale sequences.

KEY WORDS AND PHRASES: Complete convergence, L^p -mixingale sequence, q -smooth Banach space.

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1. INTRODUCTION AND MAIN RESULTS

Since the definition of complete convergence for real random variables was introduced by Hsu and Robbins[1], there have been an extensive literature in the complete convergence for independent and dependent random sequences, see partially the references listed. In particular, Yang[5,6] has discussed the complete convergence for B -valued independent random elements. Yu[11] has considered the complete convergence for martingale difference sequences, Peligrad[7] and Shao[8] have obtained the complete convergence for ϕ -mixing sequences, respectively. However, to our best acknowledgement, there are still few articles on the complete convergence for L^p -mixingale ($1 \leq p < 2$) sequences, which include uniformly mixing (called also ϕ -mixing) sequences, martingale difference sequences, linear processes and other random sequences (see [10]). In this paper, under weaker conditions of probability, we discuss the complete convergence for the partial sums and the randomly indexed partial sums of B -valued L^p -mixingale sequences, and give the complete convergence for B -valued martingale difference sequences as corollary. The methods used here are different from those used in the literature.

Next, let us introduce some notations.

Let B be a real Banach space. B is said to be q -smooth ($1 < q \leq 2$) if there exists a constant $C_q > 0$ such that for every B -valued L^q -integrable martingale difference sequence $\{D_i; i \geq 1\}$

$$E \left\| \sum_{i=1}^n D_i \right\|^q \leq C_q \sum_{i=1}^n E \|D_i\|^q, \quad n \geq 1.$$

Let $\{X_n, n \geq 1\}$ be a sequence of B -valued L^p -integrable ($1 \leq p \leq 2$) random variables on a probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}_n, -\infty < n < \infty\}$ be an increasing sequence of sub σ -fields of $\mathcal{F}$. Then $\{X_n, \mathcal{F}_n\}$ is called a L^p -mixingale sequence if there exist sequences of nonnegative constants C_n and $\psi(n)$, where $\psi(m) \downarrow 0$ as $m \rightarrow \infty$, which satisfy following properties:

$$(i) \|E(X_n|\mathcal{F}_{n-m})\|_p \leq \psi(m)C_n \quad \text{and}$$

$$(ii) \|X_n - E(X_n|\mathcal{F}_{n+m})\|_p \leq \psi(m+1)C_n$$

for all $n \geq 1$ and $m \geq 0$, where $\|X\|_p = (E\|X\|^p)^{1/p}$.

S is the class of all positive non-decreasing function ϕ on $R^+ = [0, \infty)$ (see [9], p.228 or [5,6]) satisfying the following conditions:

(i) There exists a constant $k = k(\phi) > 0$ such that

$$\phi(xy) \leq k(\phi(x) + \phi(y)), \quad \forall x, y \in R^+.$$

(ii) $x/\phi(x)$ is non-decreasing for sufficiently large x .

From now on, we will use C to denote finite positive constants whose value may change from statement to statement. For real number x , $[x]$ denote the largest integer $k \leq x$. $I(A)$ represent indicative function of set A . Put $S_n = \sum_{i=1}^n X_i$.

THEOREM 1.1. Let $1 \leq t < q \leq 2$, $0 < \delta < \frac{q}{t} - 1$, $1 \leq p \leq 2$, $d = 1$ or -1 , and let B be a q -smooth Banach space. Suppose $\{X_n, \mathcal{F}_n\}$ is a B -valued L^p -mixingale sequence, $\phi(x) \in S$. If

$$\sum_{i=1}^n P(\|X_i\|^t (\phi(\|X_i\|))^{-d} > x) \leq Cn x^{-(1+\delta)} \quad (1.1)$$

for sufficiently large x, n and there exists a $\lambda (1 \leq \lambda \leq p)$ such that $t + (1-t)\lambda > 0$ and

$$\sum_{n=1}^{\infty} \psi^\lambda([n^\beta]) \max_{1 \leq i \leq n} C_i^\lambda < \infty, \quad (1.2)$$

where $0 < \beta < \frac{\delta}{1+\delta} \wedge \frac{q-t}{t(1+\delta)}$, then for every $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1 \leq k \leq n} \|S_k\| \geq \epsilon(n(\phi(n))^d)^{1/t}) < \infty, \quad (1.3)$$

in particular

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_n\| \geq \epsilon(n(\phi(n))^d)^{1/t}) < \infty. \quad (1.4)$$

THEOREM 1.2. Under the assumptions of THEOREM 1.1, if there exists a $\lambda (1 \leq \lambda \leq p)$ satisfying

$$\sum_{m=1}^{\infty} m 2^{\lambda m} \psi^\lambda([2^{\beta m}]) \max_{1 \leq i \leq 2^{m+1}} C_i^\lambda < \infty, \quad (1.5)$$

then for every $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\sup_{k \geq n} (\|S_k\| / (k(\phi(k))^d)^{1/t}) \geq \epsilon) < \infty. \quad (1.6)$$

If $\{X_n, \mathcal{F}_n, n \geq 1\}$ is a L^p -integrable B -valued martingale difference sequence, then $C_n = (E\|X_n\|^p)^{1/p}$, $\psi(m) = 0$ for $m \geq 1$.

COROLLARY 1.1. Let $1 \leq t < q \leq 2$, $d = 1$ or -1 , and let B be a q -smooth Banach space. Suppose $\{X_n, \mathcal{F}_n, n \geq 1\}$ is a B -valued martingale difference sequence, $\phi(x) \in S$. For $0 < \delta < \frac{q}{t} - 1$ and sufficiently large x and n , if (1.1) is satisfied, then for every $\epsilon > 0$, we obtain that (1.3), (1.4) and (1.6) hold.

REMARK 1.1. For $0 < t < 1$, by C_r -inequality and properties of $\phi(x)$, we can prove that the results of THEOREM 1.1, THEOREM 1.2 and COROLLARY 1.1 hold for any B -valued random variable sequence $\{X_n, n \geq 1\}$ without mixing condition (1.2) and (1.5).

REMARK 1.2. Real uniformly mixing sequence (definition see [7] or [8], [10]) $\{X_n, \mathcal{F}_n\}$ is L^2 -mixingale, where $C_n = 2(E X_n^2)^{1/2}$, $\phi(m) = \phi^{1/2}(m)$, see [10, p.19].

REMARK 1.3. Yang[5] has proved that (1.3) and (1.4) hold for B -valued independent zero mean random element sequence $\{X_n\}$ in type 2 Banach space under moment conditions stronger than the conditions of COROLLARY 1.1.

2. PROOFS OF MAIN RESULTS

We only prove the case in $d = 1$ for Theorem 1.1 and 1.2, the proof of the case in $d = -1$ is analogous.

LEMMA 2.1.([9], Lemma 1) Let $\phi(\cdot) \in S, \delta > 0$, then for any $x \geq 0$,

$$C\phi(x) \leq \phi(x\phi(x)) \leq C\phi(x);$$

$$C\phi(x) \leq \phi(x/\phi(x)) \leq C\phi(x);$$

$$C\phi(x) \leq \phi(x^\delta) \leq C\phi(x).$$

PROOF OF THEOREM 1.1. Notice first

$$\begin{aligned} S_k &= \sum_{i=1}^k (X_i - E(X_i | \mathcal{F}_{i+[n^\theta]})) \\ &\quad + \sum_{l=-[n^\theta]+1}^{[n^\theta]} \sum_{i=1}^k (E(X_i | \mathcal{F}_{i+l}) - E(X_i | \mathcal{F}_{i+l-1})) \\ &\quad + \sum_{i=1}^k E(X_i | \mathcal{F}_{i-[n^\theta]}) \\ &\triangleq A_k^1 + B_k^1 + C_k^1. \end{aligned}$$

Obviously

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1 \leq k \leq n} \|S_k\| \geq \epsilon(n\phi(n))^{1/t}) \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1 \leq k \leq n} \|A_k^1\| \geq \frac{\epsilon}{3}(n\phi(n))^{1/t}) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1 \leq k \leq n} \|B_k^1\| \geq \frac{\epsilon}{3}(n\phi(n))^{1/t}) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1 \leq k \leq n} \|C_k^1\| \geq \frac{\epsilon}{3}(n\phi(n))^{1/t}) \\ &\triangleq I_1 + I_2 + I_3. \end{aligned}$$

By the Markovian inequality, L^p -mixingale property and the properties of $\phi(x)$, we have

$$\begin{aligned} I_1 &\leq C \sum_{n=1}^{\infty} \frac{1}{n} (n\phi(n))^{-\lambda/t} E(\sum_{i=1}^n \|X_i - E(X_i | \mathcal{F}_{i+[n^\theta]})\|)^\lambda \\ &\leq C \sum_{n=1}^{\infty} \frac{1}{n} (n\phi(n))^{-\lambda/t} (\sum_{i=1}^n \|X_i - E(X_i | \mathcal{F}_{i+[n^\theta]})\|_\lambda)^\lambda \\ &\leq C \sum_{n=1}^{\infty} (\phi(n))^{-\lambda/t} \cdot n^{\lambda-1-\lambda/t} \psi^\lambda([n^\theta]) \max_{1 \leq i \leq n} C_i^\lambda \\ &\leq C \sum_{n=1}^{\infty} \psi^\lambda([n^\theta]) \max_{1 \leq i \leq n} C_i^\lambda < \infty. \end{aligned}$$

Similarly, we can obtain

$$I_3 \leq C \sum_{n=1}^{\infty} \psi^\lambda([n^\theta]) \max_{1 \leq i \leq n} C_i^\lambda < \infty.$$

Let $Y_{i,n} = X_i I(\|X_i\| \leq (n\phi(n))^{1/t})$, $Z_{i,n} = X_i - Y_{i,n}$, $W_{i,l} = E(X_i | \mathcal{F}_{i+l}) - E(X_i | \mathcal{F}_{i+l-1})$, $U_{l,i} = E(Y_{i,n} | \mathcal{F}_{i+l}) - E(Y_{i,n} | \mathcal{F}_{i+l-1})$, $V_{l,i} = E(Z_{i,n} | \mathcal{F}_{i+l}) - E(Z_{i,n} | \mathcal{F}_{i+l-1})$, $1 \leq k \leq n$, $1 \leq i \leq k$, $-[n^\theta] + 1 \leq l < [n^\theta]$.

Clearly, $X_i = Y_{i,n} + Z_{i,n}$, $W_{l,i} = U_{l,i} + V_{l,i}$. For fixed l , $\{U_{l,i}, \mathcal{F}_{i+l}, 1 \leq i \leq n\}$ and $\{V_{l,i}, \mathcal{F}_{i+l}, 1 \leq$

$\{ \leq n \}$ are martingale difference sequences. Then

$$\begin{aligned} I_2 &\leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{l=-[n^{\beta}]+1}^{[n^{\beta}]} P(\max_{1 \leq k \leq n} \| \sum_{i=1}^k W_{l,i} \| \geq \frac{\epsilon}{6} (n\phi(n))^{1/t} \cdot n^{-\beta}) \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{l=-[n^{\beta}]+1}^{[n^{\beta}]} P(\max_{1 \leq k \leq n} \| \sum_{i=1}^k U_{l,i} \| \geq \frac{\epsilon}{12} (n\phi(n))^{1/t} \cdot n^{-\beta}) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n} \sum_{l=-[n^{\beta}]+1}^{[n^{\beta}]} P(\max_{1 \leq k \leq n} \| \sum_{i=1}^k V_{l,i} \| \geq \frac{\epsilon}{12} (n\phi(n))^{1/t} \cdot n^{-\beta}) \\ &\triangleq I_4 + I_5. \end{aligned}$$

Since B is q -smoothable, therefore using Doob inequality, the monotone property of $x/\phi(x)$ and Lemma 2.1 we have

$$\begin{aligned} I_4 &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \sum_{l=-[n^{\beta}]+1}^{[n^{\beta}]} (n\phi(n))^{-q/t} \cdot n^{\beta q} \sum_{i=1}^n E \|Y_{i,n}\|^q \\ &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \cdot [n^{\beta}] \cdot (n\phi(n))^{-q/t} \cdot n^{\beta q} \sum_{i=1}^n \int_0^{n\phi(n)} x^{q/t-1} P(\|X_i\|^t > x) dx \\ &\leq C \sum_{n=1}^{\infty} (\phi(n))^{-q/t} \cdot n^{-[q/t-(1+q)\beta]} \\ &\quad + C \sum_{n=1}^{\infty} (\phi(n))^{-q/t} \cdot n^{-[q/t-(1+q)\beta]-1} \int_C^{n\phi(n)} x^{q/t-1} \sum_{i=1}^n P(\|X_i\|^t / \phi(\|X_i\|) \geq Cx / \phi(x)) dx \\ &\leq C + C \sum_{n=1}^{\infty} n^{(1+q)\beta-(1+\delta)} < \infty. \end{aligned}$$

By applying the definition of $\phi(x)$ and Lemma 2.1, we can obtain

$$\phi(x^{\alpha}) \leq Cx^{\beta} \quad (2.1)$$

for $\alpha, \beta > 0$ and sufficiently large x .

By the Markovian inequality, the definition of $\phi(x)$, Lemma 2.1 and (2.1) we have

$$\begin{aligned} I_5 &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \sum_{l=-[n^{\beta}]+1}^{[n^{\beta}]} (n\phi(n))^{-1/t} \cdot n^{\beta} \sum_{i=1}^n E \|Z_{i,n}\| \\ &\leq C \sum_{n=1}^{\infty} \frac{1}{n} \cdot [n^{\beta}] \cdot (n\phi(n))^{-1/t} \cdot n^{\beta} \sum_{i=1}^n [(n\phi(n))^{1/t} P(\|X_i\| > (n\phi(n))^{1/t}) \\ &\quad + \int_{(n\phi(n))^{1/t}}^{\infty} P(\|X_i\| > x) dx] \\ &\leq C \sum_{n=1}^{\infty} n^{2\beta-1} \sum_{i=1}^n [P(\|X_i\|^t / \phi(\|X_i\|) \geq Cn) + \int_1^{\infty} P(\|X_i\| > y(n\phi(n))^{1/t}) dy] \\ &\leq C \sum_{n=1}^{\infty} n^{2\beta-(1+\delta)} + C \sum_{n=1}^{\infty} n^{2\beta-1} \int_1^{\infty} \sum_{i=1}^n P(\frac{\|X_i\|^t}{\phi(\|X_i\|)} \geq \frac{C y^t (n\phi(n))}{\phi(y^t) + C \phi(n)}) dy \\ &\leq C + C \sum_{n=1}^{\infty} n^{2\beta-(1+\delta)} \int_1^{\infty} [\frac{(\phi(y^t))^{1+\delta}}{y^{t(1+\delta)}} + \frac{1}{y^{t(1+\delta)}}] dy < \infty. \end{aligned}$$

The proof is completed.

PROOF OF THEOREM 1.2. First, by the monotone property of $\phi(x)$ we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n} P(\sup_{k \geq n} \|S_k\| / (k\phi(k))^{1/t} \geq \epsilon) \\ &\leq \sum_{i=0}^{\infty} P(\sup_{k \geq 2^i} \|S_k\| / (k\phi(k))^{1/t} \geq \epsilon) \\ &\leq C \sum_{m=1}^{\infty} m P(\max_{2^m \leq k < 2^{m+1}} (\|S_k\| \geq \epsilon(2^m \phi(2^m))^{1/t}). \end{aligned} \quad (2.2)$$

Observe that for $2^m \leq k < 2^{m+1}$,

$$\begin{aligned} S_k &= \sum_{i=1}^k (X_i - E(X_i | \mathcal{F}_{i+[2^m]})) \\ &\quad + \sum_{l=-[2^m]+1}^{[2^m]} \sum_{i=1}^k (E(X_i | \mathcal{F}_{i+l}) - E(X_i | \mathcal{F}_{i+l-1})) \\ &\quad + \sum_{i=1}^k E(X_i | \mathcal{F}_{i-[2^m]}) \\ &\triangleq A_k^2 + B_k^2 + C_k^2. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n} P(\sup_{k \geq n} \|S_k\| / (k\phi(k))^{1/t} \geq \epsilon) \\ & \leq C \sum_{m=1}^{\infty} m P\left(\max_{2^m \leq k < 2^{m+1}} (\|A_k^2\| \geq \frac{\epsilon}{3} (2^m \phi(2^m))^{1/t})\right) \\ & \quad + C \sum_{m=1}^{\infty} m P\left(\max_{2^m \leq k < 2^{m+1}} (\|B_k^2\| \geq \frac{\epsilon}{3} (2^m \phi(2^m))^{1/t})\right) \\ & \quad + C \sum_{m=1}^{\infty} m P\left(\max_{2^m \leq k < 2^{m+1}} (\|C_k^2\| \geq \frac{\epsilon}{3} (2^m \phi(2^m))^{1/t})\right) \\ & \triangleq I_6 + I_7 + I_8. \end{aligned}$$

By analogizing the proof of I_1 we have

$$I_6 \leq C \sum_{m=1}^{\infty} m 2^{\lambda m} \psi^{\lambda}([2^{\beta m}]) \max_{1 \leq i \leq 2^{m+1}} C_i^{\lambda} < \infty.$$

By analogizing the proof of I_3 , similarly, we can obtain $I_8 < \infty$.

Let $Y_{i,m} = X_i I(\|X_i\| \leq (2^m \phi(2^m))^{1/t})$, $Z_{i,m} = X_i - Y_{i,m}$, $W_{l,i} = E(X_i | \mathcal{F}_{i+l}) - E(X_i | \mathcal{F}_{i+l-1})$, $U_{l,i} = E(Y_{i,m} | \mathcal{F}_{i+l}) - E(Y_{i,m} | \mathcal{F}_{i+l-1})$, $V_{l,i} = E(Z_{i,m} | \mathcal{F}_{i+l}) - E(Z_{i,m} | \mathcal{F}_{i+l-1})$, $2^m \leq k < 2^{m+1}$, $1 \leq i \leq k$, $-[2^{\beta m}] + 1 \leq l \leq [2^{\beta m}]$. Then

$$\begin{aligned} I_7 & \leq C \sum_{m=1}^{\infty} m \sum_{l=-[2^{\beta m}]+1}^{[2^{\beta m}]} P\left(\max_{2^m \leq k < 2^{m+1}} \left\| \sum_{i=1}^k U_{l,i} \right\| \geq \frac{\epsilon}{12} (2^m \phi(2^m))^{1/t} \cdot 2^{-\beta m}\right) \\ & \quad + C \sum_{m=1}^{\infty} m \sum_{l=-[2^{\beta m}]+1}^{[2^{\beta m}]} P\left(\max_{2^m \leq k < 2^{m+1}} \left\| \sum_{i=1}^k V_{l,i} \right\| \geq \frac{\epsilon}{12} (2^m \phi(2^m))^{1/t} \cdot 2^{-\beta m}\right) \\ & \triangleq I_9 + I_{10}. \end{aligned}$$

By analogizing the proof of I_4 we have

$$I_9 \leq C \sum_{m=1}^{\infty} m 2^{(\beta+q\beta+1-q/t)m} (\phi(2^m))^{-q/t} + C \sum_{m=1}^{\infty} m 2^{(\beta+q\beta-\delta)m} < \infty.$$

By analogizing the proof of I_5 we have

$$I_{10} \leq C \sum_{m=1}^{\infty} m 2^{(2\beta-\delta)m} + C \sum_{m=1}^{\infty} m 2^{(2\beta-\delta)m} \int_1^{\infty} \left| \frac{(\phi(y'))^{1+\delta}}{y^{t(1+\delta)}} + \frac{1}{y^{t(1+\delta)}} \right| dy < \infty.$$

3. RANDOMLY INDEXED PARTIAL SUMS

Throughout this section let $\{X_n, \mathcal{F}_n\}$ be a B -valued L^p -mixingale sequence ($1 \leq p \leq 2$), and let $\{\tau_n, n \geq 1\}$ be a sequence of nonnegative, integer valued random variables. τ is a positive random variable. All random variables are defined on the same probability space.

THEOREM 3.1. Under the assumptions of THEOREM 1.2, if there exists some constant $\epsilon_0 > 0$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\frac{\tau_n}{n} < \epsilon_0\right) < \infty, \quad (3.1)$$

then for every $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_{\tau_n}\| \geq \epsilon (\tau_n (\phi(\tau_n))^d)^{1/t}) < \infty. \quad (3.2)$$

THEOREM 3.2. Under the assumptions of THEOREM 1.1, if there exist constants a, b, ϵ_0 ($0 < \epsilon_0 < a \leq b < \infty$) such that $P(a \leq \tau \leq b) = 1$ and

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left|\frac{\tau_n}{n} - \tau\right| > \epsilon_0\right) < \infty, \quad (3.3)$$

then for every $\epsilon > 0$ we obtain that (3.2) holds.

THEOREM 3.3. Under the assumptions of THEOREM 1.1, if there exist constants $b > 0$ and $\epsilon_0 > 0$ such that $P(\tau \leq b) = 1$ and (3.3) is satisfied, then for every $\epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_n\| \geq \epsilon(n(\phi(n))^d)^{1/t}) < \infty.$$

Obviously, suppose $P(\tau \geq a) = 1$ for some $a > 0$, then for any $\epsilon > 0$ ($\epsilon < a$) we have

$$P\left(\frac{\tau_n}{n} < a - \epsilon\right) \leq P\left(\left|\frac{\tau_n}{n} - \tau\right| > \epsilon\right),$$

therefore, if condition (3.3), where $P(\tau \geq a) = 1$ for some $a > \epsilon_0 > 0$ replaces condition (3.1), then THEOREM 3.1 still holds.

Similarly, using COROLLARY 1.1, we can obtain the complete convergence for the randomly indexed partial sums of B -valued martingale difference sequences, respectively.

REMARK 3.1. Condition (3.1) and (3.3) are just ones which are usually employed in literature.

REMARK 3.2. Note that if $\phi(x) = 1$, $L_k(x) (L_0(x) = \max(1, \log x), L_k = \log[\max(e, L_{k-1}(x))]$, $k = 1, 2, \dots$), we can derive many significative results from the results of this paper. In addition, since real space is a 2-smooth Banach space, the THEOREM and COROLLARY in this paper are suitable for real valued random variable.

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