ON STRICT AND SIMPLE TYPE EXTENSIONS

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ABSTRACT. Let \((Y, \tau)\) be an extension of a space \((X, \tau')\). \(p \in Y\), let \(\mathcal{O}_p^\tau = \{W \cap X : W \in \tau, p \in W\}\). For \(U \in \tau'\), let \(o(U) = \{p \in Y : U \cap Y, p \in W\}\). In 1964, Banaschewski introduced the strict extension \(Y^\#\), and the simple extension \(Y^+\) of \(X\) (induced by \((Y, \tau)\)) having base \(\{o(U) : U \in \tau'\}\) and \(\{W : p \in Y, U \in \mathcal{O}_p^\tau\}\), respectively. The extensions \(Y^\#\) and \(Y^+\) have been extensively used since then. In this paper, the open filters \(\mathcal{L}^\tau = \{W \in \tau : \text{int}_X \text{cl}_X(U) \subseteq \text{int}_X \text{cl}_X(W)\}\) for some \(U \in \mathcal{O}_p^\tau\), and \(U^\tau = \{W \in \tau : \text{int}_X \text{cl}_X(W) \in \mathcal{O}_p^\tau\}\) are used to define some new topologies on \(Y\). Some of these topologies produce nice extensions of \((X, \tau')\). We study some interrelationships of these extensions with \(Y^\#\), and \(Y^+\) respectively.

KEY WORDS AND PHRASES: Extension, simple extension, strict extension, H-closed, s-closed, almost realcompact, near compact.

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1. INTRODUCTION

A topological space \(Y\) is an extension of a space \(X\) if \(X\) is a dense subspace of \(Y\). If \(Y_1\) and \(Y_2\) are two extensions of a space \(X\), then \(Y_2\) is said to be projectively larger than \(Y_1\), written \(Y_2 \geq Y_1\) (or \(Y_1 \leq Y_2\)), provided that there exists a continuous map \(f : Y_2 \to Y_1\) such that \(f|_X = i_X\), the identity map on \(X\). Two extensions \(Y_1\) and \(Y_2\) of \(X\) are called equivalent if \(Y_1 \leq Y_2\) and \(Y_2 \leq Y_1\). We shall identify two equivalent extensions of \(X\). With this convention, the class \(E(X)\) of all the Hausdorff extensions of a Hausdorff space \(X\) is a set. Let \((Y, \tau) \in E(X)\) and let \(p \in Y\). If \(N_p\) is the open neighborhood filter of \(p\) in \(Y\), the set \(\mathcal{O}_p^\tau = \{N \cap X : N \in N_p\}\) (called the trace of \(N_p\) on \(X\)) is an open filter on \(X\). If \(U\) is open in \(X\), denote \(o_U(U) = \{p \in Y : U \in \mathcal{O}_p^\tau\}\).
In 1964 Banaschewski [1] introduced the extensions \( Y^* \) (resp. \( Y^+ \)) the strict extension (resp. the simple extension) of \( X \) induced by \( Y \) satisfying \( Y^* \leq Y \leq Y^+ \). The topology \( \tau^* \) on \( Y^* \) (resp. \( \tau^+ \) on \( Y^+ \)) has for an open base the collection \( \{ o, (U): U \text{ open in } X \} \) (resp., the collection \( \{ U \cup \{ p \}: p \in Y, \text{ and } U \in \mathcal{F}_p \} \)). The extensions \( Y^* \) and \( Y^+ \) have been studied extensively and have proved extremely useful regarding some properties weaker than compactness, such as nearly compact, almost realcompact, feebly compact, \( H \)-closed, \( s \)-closed, etc. In this paper we introduce new extensions \( Y', Y'', Y'^*, \) and \( Y'^* \), study some of their properties, and compare them with \( Y, Y^*, \) and \( Y^+ \). All spaces under consideration are Hausdorff.

2. THE EXTENSIONS \( Y' \) AND \( Y'' \).

In this section, we introduce several topologies on \( Y \), and compare them with \( \tau \). Some of these topologies yield interesting extensions of \( (X, \tau') \).

**DEFINITION 2.1.** Let \((Y, \tau)\) be an extension of a space \((X, \tau')\). For \( p \in Y \) define

\[
U^p = \{ W: W \in \tau', \text{int}_x \text{cl}_x W \in \mathcal{F}_p \}, \tag{2.1}
\]

\[
\mathcal{L}^p = \{ W: W \in \tau', W \supseteq \text{int}_x \text{cl}_x U \text{ for some } U \in \mathcal{F}_p \}. \tag{2.2}
\]

**LEMMA 2.1.**

(a) Both \( U^p \) and \( \mathcal{L}^p \) are open filters on \( X \) such that \( \mathcal{L}^p \subseteq \mathcal{F}_p \subseteq U^p \).

(b) \( U^p = \{ W: W \in \tau', \text{int}_x \text{cl}_x W \in \mathcal{L}^p \} = \cap \{ \mathcal{L}^p: \text{ an open ultrafilter on } X, \mathcal{F}_p \subseteq \mathcal{L}^p \} \)

**PROOF.** We prove (b). Let \( \mathcal{U} = \{ W: W \in \tau', \text{int}_x \text{cl}_x W \in \mathcal{L}^p \} \). If \( W \in \mathcal{U} \), then \( W \in \tau' \) and \( \text{int}_x \text{cl}_x W \supseteq \text{int}_x \text{cl}_x U \) for some \( U \in \mathcal{F}_p \). Therefore, \( \text{int}_x \text{cl}_x W \in \mathcal{F}_p \), whence \( W \in U^p \). Thus, \( \mathcal{U} \subseteq U^p \).

To prove the reverse inequality, let \( W \in U^p \). Then \( \text{int}_x \text{cl}_x W \in \mathcal{F}_p \). Since \( \text{int}_x \text{cl}_x W \supseteq \text{int}_x \text{cl}_x (\text{int}_x \text{cl}_x W) \) it follows that \( \text{int}_x \text{cl}_x W \in \mathcal{L}^p \). Hence \( W \in \mathcal{U} \). This proves the first equality in (b). The second equality follows from [9], completing the proof of the lemma.

**REMARK 2.1.** Since \( \mathcal{F}_p = \mathcal{F}_p^* = \mathcal{F}_p + [9,10,11] \), it follows that each one of \( Y, Y' \) and \( Y'' \) yield the same \( \mathcal{L}^p \) (resp., \( U^p \)) for all \( p \in Y \). Moreover, if \( Z \in E(X) \) has the same underlying set as \( Y \), and is such that \( Y^* \leq Z \leq Y^+ \), then \( Y \) and \( Z \) induce the same \( \mathcal{L}^p \) (resp., \( U^p \)) for all \( p \in Y \). Also, if \( p \neq q \) are distinct elements of \( Y \) then \( \mathcal{L}^p \neq \mathcal{L}^q \) and \( U^p \neq U^q \). Obviously, if \( U \in \mathcal{F}_p \), then \( \text{int}_x \text{cl}_x (U) \in \mathcal{L}^p \). Moreover, \( U \in U^p \) if and only if \( \text{int}_x \text{cl}_x (U) \in \mathcal{F}_p \).

**DEFINITION 2.2.** Let \((Y, \tau)\) be an extension of \((X, \tau')\). For \( G \in \tau' \), define

\[
\alpha(G) = G \cup \{ p: p \in Y \setminus X, G \in \mathcal{L}^p \} \tag{2.3}
\]
\[
oalign{\smallskip}
o_{\ast}(G) = G \cup \{ p \in Y \setminus X, G \in U \}
\]
\hspace{1.25em} \text{(2.4)}

\[
o_{\ast}(G) = \{ p \in Y : G \in U \}
\]
\hspace{1.25em} \text{(2.5)}

\[
o_{\ast}(G) = \{ p \in Y : G \in U \}
\]
\hspace{1.25em} \text{(2.6)}

The proof of the Propositions 2.1, and 2.2 is straightforward.

**Proposition 2.1.** Let \((Y, \tau)\) be an extension of \((X, \tau')\). Then for all \(U, V \in \tau'\)

(a) \(\alpha(\emptyset) = \emptyset, \alpha(X) = Y\),
(b) \(\alpha(U \cap X) = U\),
(c) \(\alpha(U \cap V) = \alpha(U) \cap \alpha(V)\),
(d) The family \(\{\alpha(G) : G \in \tau'\}\) is an open base for a Hausdorff topology \(\tau_i\) on \(Y\) and \((Y, \tau_i)\) is an extension of \(X\).

**Proposition 2.2.** Let \((Y, \tau)\) be an extension of \((X, \tau')\). Then for all \(U, V \in \tau'\),

(a) \(\alpha(\emptyset) = \emptyset\) and \(\alpha(X) = Y\),
(b) \(\alpha(U \cap X) = U\),
(c) \(\alpha(U \cap V) = \alpha(U) \cap \alpha(V)\),
(d) The family \(\{\alpha(G) : G \in \tau'\}\) is an open base for a Hausdorff topology \(\tau_u\) on \(Y\) and \((Y, \tau_u)\) is an extension of \(X\).

**Proposition 2.3.** Let \((Y, \tau)\) be an extension of \((X, \tau')\). Then for all \(U, V \in \tau'\),

(a) \(\alpha(\emptyset) = \emptyset, \alpha(X) = Y\),
(b) \(\alpha(U \cap X) \subseteq U\),
(c) \(\alpha(U \cap V) = \alpha(U) \cap \alpha(V)\),
(d) \(\alpha(U) = \cup \{ W : W \in \tau \text{ and } \text{int}_X \text{cl}_X(W \cap X) \subseteq U \}\)
(e) The family \(\{\alpha(G) : G \in \tau'\}\) is an open base for a coarser Hausdorff topology \(\tau_{\text{ul}}\) on \(Y\), \(X\) is dense in \((Y, \tau_{\text{ul}})\), but \((Y, \tau_{\text{ul}})\) may not be an extension of \(X\).

**Proof.** We prove (d). The rest is straightforward. Let \(p \in \alpha(U)\). Then \(U \in \mathcal{U}\). Therefore, \(U \supseteq \text{int}_X \text{cl}_X V\) for some \(V \in \mathcal{V}\). Therefore, there exists \(W \in \tau\) such that \(p \in W\) and \(W \cap X = V\). It follows that \(\text{int}_X \text{cl}_X(W \cap X) \subseteq U\). Conversely, if \(W \in \tau\) is such that \(\text{int}_X \text{cl}_X(W \cap X) \subseteq U\) and \(p \in W\), then \(W \cap X \in \mathcal{V}\). So, \(\text{int}_X \text{cl}_X(W \cap X) \in \mathcal{U}\). This implies that \(U \in \mathcal{U}\) and hence \(p \in \alpha(U)\). The proof of the proposition is now complete.

**Proposition 2.4.** Let \((Y, \tau)\) be an extension of \((X, \tau')\). Then for all \(U, V \in \tau'\),

(a) \(\alpha(\emptyset) = \emptyset\) and \(\alpha(X) = Y\),
(b) \( a_\circ(U) \cap X = \text{int}_x \text{cl}_x(U) \),

(c) \( a_\circ(U \cap V) = a_\circ(U) \cap a_\circ(V) \),

(d) \( a_\circ(U) = \cup \{ W : W \in \tau \text{ and } W \cap X \subseteq \text{int}_x \text{cl}_x(U) \} \)

(e) The family \( \{ a_\circ(G) : G \in \tau' \} \) is an open base for a coarser Hausdorff topology \( \tau_\circ \) on \( X \), \( X \) is dense in \( (Y, \tau_\circ) \), but \( (Y, \tau_\circ) \) may not be an extension of \( X \).

**PROOF.** We prove (d). The rest is straightforward. Let \( p \in a_\circ(U) \). Then \( U \subseteq U' \). Therefore, \( \text{int}_x \text{cl}_x U \subseteq U' \). It follows that there exists \( W \in \tau \) such that \( p \in W \) and \( W \cap X \subseteq \text{int}_x \text{cl}_x U \).

Conversely, if \( W \in \tau \) is such that \( W \cap X \subseteq \text{int}_x \text{cl}_x U \) and \( p \in W \), then \( W \cap X \subseteq U' \). So, \( \text{int}_x \text{cl}_x U \subseteq U' \). Therefore, \( U \subseteq U' \) and \( p \in a_\circ(U) \).

**DEFINITION 2.3.** The spaces \((Y, \tau_\circ), (Y, \tau_\circ'), (Y, \tau_\circ^\circ), \) and \((Y, \tau_\circ^\circ)\) described in propositions 2.1-2.4 will, henceforth, be denoted by \( Y', Y^\circ, Y'^\circ, \) and \( Y^\circ' \) respectively. If \( A \subseteq Y \), then \( \text{int}_\tau(A) \) (resp. \( \text{cl}_\tau(A) \)) will be denoted by \( \text{int}_\tau(A) \) (resp., \( \text{cl}_\tau(A) \)). Likewise, \( \text{int}_\tau(A), \text{cl}_\tau(A), \text{int}_\tau(A), \text{cl}_\tau(A), \text{int}_\tau(A) \) and \( \text{cl}_\tau(A) \) are defined in an analogous manner.

**LEMMA 2.2.** If \( U \in \tau' \), then

(a) \( a_\circ(U) \subseteq o_\circ(U) \subseteq o_\tau(U) \subseteq a_\circ(\text{int}_x \text{cl}_x U) = a_\circ(U) \cap X = a_\circ(\text{int}_x \text{cl}_x U) \),

(b) \( a_\circ(U) \cap X = o_\circ(U) \cap X \), and \( a_\circ(U) \cap X = o_\tau(U) \cap X \)

(c) \( o_\circ(\text{int}_x \text{cl}_x U) \cap X = a_\circ(U) \cap X \), and

(d) if \( U \) is regular open (i.e. \( U \subseteq \text{int}_x \text{cl}_x U \)), then \( a_\circ(U) = a_\circ(U) \), and the equality holds in (a).

**PROOF.** Part (a): We show that \( o_\circ(\text{int}_x \text{cl}_x U) = a_\circ(U) \), the rest being straightforward. Certainly, \( o_\circ(\text{int}_x \text{cl}_x U) \cap X = \text{int}_x \text{cl}_x U = a_\circ(U) \cap X \). Let \( p \in o_\circ(\text{int}_x \text{cl}_x U) \cap X \). Then \( \text{int}_x \text{cl}_x U \subseteq U' \). Therefore, \( U \subseteq U' \), and \( p \in a_\circ(U) \cap X \). Conversely, let \( p \in a_\circ(U) \cap X \). Then, \( U \subseteq U' \). So, \( p \in o_\circ(U) \cap X \subseteq o_\circ(\text{int}_x \text{cl}_x U) \cap X \). The above arguments prove (a).

To prove (c), let \( q \in o_\circ(\text{int}_x \text{cl}_x G) \cap X \). Then, \( \text{int}_x \text{cl}_x G \subseteq U' \). Therefore, \( q \in o_\circ(G) \cap X \). Thus, \( o_\circ(\text{int}_x \text{cl}_x G) \cap X = o_\circ(G) \cap X \). To prove the reverse inequality, let \( q \in o_\circ(G) \cap X \). Then, \( G \subseteq U' \). Therefore, \( q \in o_\circ(\text{int}_x \text{cl}_x G) \cap X \) and \( o_\circ(G) \cap X \subseteq o_\circ(\text{int}_x \text{cl}_x G) \cap X \). Hence, \( o_\circ(\text{int}_x \text{cl}_x G) \cap X = o_\circ(G) \cap X \). The rest of the lemma is straightforward.

Given a space \((X, \tau')\), the family \( \{ \text{int}_x \text{cl}_x U : U \in \tau' \} \) forms an open base for a coarser Hausdorff topology \( \tau'_1 \) on \( X \). The space \( X' = (X, \tau'_1) \) is called the *semiregularization* of \( X \). A space \((X, \tau')\) is called *semiregular* if \((X, \tau') = X\).

**THEOREM 2.1.** If \( X \) is semiregular, and \((Y, \tau)\) (not necessarily semiregular) is an extension of \( X \), then \( Y' \) is an extension of \( X \) such that \( Y' \subseteq Y \).

**PROOF.** If \( X \) is semiregular, then \( o_\circ(U) = a_\circ(U) \) for all \( U \in \tau' \). Hence, \( Y' \) is an extension of \( X \) such that \( Y' \subseteq Y' \subseteq Y \).
THEOREM 2.2. The spaces $Y'$ and $Y''$ are homeomorphic.

PROOF. For all $U \in \tau'$, $a_\tau(\text{int}_x \text{cl}_y U) = a_\tau(\text{int}_x \text{cl}_y U) = a_\tau(U)$ implies that $\tau' \subseteq \tau''$. Also, if $G \in \tau'$ and $p \in a_\tau(G)$, then $G \supseteq \text{int}_x \text{cl}_y U$ for some $U \in \tau_y \subseteq \tau''$. Now, if $q \in a_\tau(U)$, then $\text{int}_x \text{cl}_y U \in \tau''$ which implies that $G \in \tau''$, or $q \in a_\tau(G)$. Therefore, $p \in a_\tau(U) \subseteq a_\tau(G)$. Hence, $\tau' \subseteq \tau''$. This proves the theorem.

LEMMA 2.3. Let $(Y, \tau)$ be an extension of $(X, \tau')$. Then, for all $G \in \tau'$ the following are true.

(a) $\text{cl}_\tau(G) \subseteq \text{cl}_\tau(\text{int}_x \text{cl}_y(G))$
(b) $\text{cl}_\tau(G) = \text{cl}_\tau(\text{int}_x \text{cl}(G))$
(c) $\text{cl}_\tau(G) = \text{cl}_\tau(G)$
(d) $\text{cl}_\tau(G) = \text{cl}_\tau(G) = \text{cl}_\tau(\text{int}_x \text{cl}(G))$, and
(e) $\text{cl}_\tau(\text{int}_x(G)) = \text{cl}_\tau(\text{int}_x \text{cl}(G))$

PROOF. Part (a): Let $p \in \text{cl}_\tau(G)$, and let $o_\tau(U)$ be a basic open neighborhood of $p$ in $Y$. If $G \subseteq o_\tau(U) \cap X$, then $p \in U \subseteq \text{int}_x \text{cl}_y U \subseteq \tau''$. Therefore, $a_\tau(\text{int}_x \text{cl}_y U)$ is an open neighborhood of $p$ in $Y''$. Consequently, $a_\tau(\text{int}_x \text{cl}_y U) \cap G \neq \emptyset$. By Proposition (2.7) (b), $\text{int}_x \text{cl}_y U \cap G \neq \emptyset$. Hence $U \cap G \neq \emptyset$. This in turn implies that $o_\tau(U) \cap G \neq \emptyset$, and $p \in \text{cl}_\tau(G)$. If $p \in o_\tau(U) \setminus X$, then $U \in \tau''$. Now, $a_\tau(G)$ is an open neighborhood of $p$ in $Y''$. Consequently, $a_\tau(U) \cap G \neq \emptyset$. Therefore, $o_\tau(U) \cap G \neq \emptyset$ whence $p \in \text{cl}_\tau(G)$.

Part (b): Let $p \in \text{cl}_\tau(G)$, and let $o_\tau(U)$ be a basic open neighborhood of $p$ in $Y$. Since $o_\tau(U) \subseteq a_\tau(G)$, $a_\tau(U)$ is an open neighborhood of $p$ in $Y''$. Hence, $a_\tau(U) \cap G \neq \emptyset$. Therefore, $\text{int}_x \text{cl}_y U \cap G \neq \emptyset$, whence $U \cap G \neq \emptyset$. Consequently, $a_\tau(U) \cap G \neq \emptyset$. Therefore, $p \in \text{cl}_\tau(G)$. The other half of (b) is straightforward.

The proof of (c) is straightforward.

Part (d): Let $p \in \text{cl}_\tau(G)$, and let $W$ be an open neighborhood of $p$ in $Y$. Then, $W \cap X \in \tau_y \subseteq \tau''$ shows that $o_\tau(W \cap X)$ is an open neighborhood of $p$ in $Y''$. Therefore, $a_\tau(W \cap X) \neq \emptyset$. This shows that $W \cap G \neq \emptyset$, whence $p \in \text{cl}_\tau(G)$. Conversely, let $p \in \text{cl}_\tau(G)$, and let $a_\tau(U)$ be a basic open neighborhood of $p$ in $Y''$. Then, $U \in \tau''$. So, $o_\tau(\text{int}_x \text{cl}_y U)$ is an open neighborhood of $p$ in $Y$ such that $o_\tau(\text{int}_x \text{cl}_y U) \cap G \neq \emptyset$. This implies that $a_\tau(U) \cap G \neq \emptyset$. Hence, $p \in \text{cl}_\tau(G)$. The rest follows from (c).

THEOREM 2.3. The spaces $Y' \setminus X, Y'' \setminus X$, and $Y'' \setminus X$ are pairwise homeomorphic.
PROOF. To prove the continuity of the identity map \( i: Y^\omega \setminus X \to Y^\iota \setminus X \), let \( o_i(G) \setminus X \) be a basic open neighborhood of \( p \) in \( Y^\iota \setminus X \). Then, \( G \in \mathcal{F}^\iota \). Hence \( G \supseteq \text{int}_x \text{cl}_x U \) for some \( U \in \mathcal{G}^\iota \subseteq \mathcal{F}^\iota \). Therefore, \( o_i(U) \setminus X \) is an open neighborhood of \( p \) in \( Y^\omega \) such that \( o_i(U) \setminus X \subseteq o_i(G) \setminus X \). To prove that the identity map \( i: Y^\iota \setminus X \to Y^\iota \setminus X \) is continuous, let \( o_i(G) \setminus X \) be a basic open neighborhood of \( p \) in \( Y^\omega \setminus X \). Then \( o_i(\text{int}_x \text{cl}_x G) \setminus X \) is an open neighborhood of \( p \) in \( Y^\omega \) such that \( o_i(\text{int}_x \text{cl}_x G) \setminus X = o_i(G) \setminus X \). Hence, the spaces \( Y^\iota \setminus X \), and \( Y^\omega \setminus X \) are homeomorphic. The rest of the theorem follows directly from Lemma 2.2.

Let \( Z_1 \) and \( Z_2 \) be spaces. A map \( f: Z_1 \to Z_2 \) is called \( \theta \)-continuous [3] if for every \( p \in Z_1 \) and for every open neighborhood \( V \) of \( f(p) \) in \( Z_2 \), there exists an open neighborhood \( U \) of \( p \) in \( Z_1 \) such that \( f(\text{cl}_z U) \subseteq \text{cl}_z (V) \). \( f \) is called perfect if \( f \) is a closed map (not necessarily continuous) such that \( f^{-1}(z) \) is compact in \( Z_1 \) for every \( z \in Z_2 \). Also, \( f \) is called irreducible if \( f \) is closed and there is no proper closed subset \( K \) of \( Z_1 \) for which \( f(K) = Z_2 \). Two extensions \( Z_1 \), and \( Z_2 \) of a space \( X \) are called \( \theta \)-equivalent if there exists a \( \theta \)-homeomorphism \( f \) from \( Z_1 \) onto \( Z_2 \) such that \( f|_X = i_X \), the identity map on \( X \).

The next theorem depicts some of the several interrelationships between the spaces \( Y, Y^\iota, Y^\omega, \) and \( Y^\omega \).

**THEOREM 2.4.** Let \( (Y, \tau) \) be an extension of a space \( (X, \tau') \). The following statements are true.

(a) The identity map \( i: Y^\omega \to Y \) is perfect, irreducible and \( \theta \)-continuous.

(b) The identity map \( i: Y^\omega \to Y^\iota \) is perfect, irreducible and \( \theta \)-continuous.

(c) The identity map \( i: Y^\omega \to Y^\iota \) is \( \theta \)-continuous.

(d) The identity map \( i: Y^\iota \to Y^\iota \) is \( \theta \)-continuous.

(e) The identity map \( i: Y^\iota \to Y^\iota \) is \( \theta \)-continuous.

(f) The identity map \( i: Y^\iota \to Y^\iota \) is \( \theta \)-continuous.

(g) The identity map \( i: Y^\iota \to Y^\iota \) is \( \theta \)-continuous.

(h) The identity map \( i: Y^\iota \to Y^\iota \) is \( \theta \)-continuous.

(i) The identity map \( i: Y^\iota \to Y^\iota \) is \( \theta \)-continuous.

(j) The identity map \( i: Y^\iota \to Y^\iota \) is \( \theta \)-continuous.

(k) The identity map \( i: Y^\iota \to Y^\iota \) is \( \theta \)-continuous.

(l) The identity map \( i: Y^\iota \to Y^\iota \) is \( \theta \)-continuous.

PROOF. Below, we outline the proofs of some parts of the theorem. The rest of the proofs are analogous.

Part (a) Since \( \tau_\omega \leq \tau, i: Y \to Y^\omega \) is continuous. Hence, \( i: Y \to Y^\omega \) is irreducible and perfect. To prove the \( \theta \)-continuity of \( i: Y^\omega \to Y \), let \( V \) be an open neighborhood of \( p \) in \( Y \). Then \( V \cap X \in \mathcal{G}^\omega \) and \( \text{int}_x \text{cl}_x (V \cap X) \in \mathcal{F}^\omega \). Therefore, \( a_i(\text{int}_x \text{cl}_x (V \cap X)) \) is an open neighborhood of \( p \) in \( Y^\omega \) such that
Proposition 2.5. Let $\phi$ and $\psi$ be two $\theta$-equivalent extensions of a space $X$, and let $X'$ be a $\theta$-extension of $X$. Then, $\phi$ and $\psi$ are $\theta$-homeomorphic if and only if $X'$ is dense in $\phi$ and $\psi$. If $X'$ is not dense in both $\phi$ and $\psi$, then $\phi$ and $\psi$ are not $\theta$-homeomorphic.

Corollary 2.1. Let $(Y,\tau)$ be an extension of $(X,\tau')$, and let $Y''$ be a $\theta$-extension of $Y$. Then, $Y''$ is $\theta$-equivalent to $Y$. Moreover, $Y''$ is a $\theta$-extension of $Y$ if and only if $Y'$ is a $\theta$-extension of $Y$. If $Y''$ is not a $\theta$-extension of $Y$, then $Y''$ is not $\theta$-equivalent to $Y$.
LEMMA 3.1. For each $G \in \tau'$, $cl_\omega(o(G)) = cl_\omega(o(G))$ and $cl_\omega(o(G)) = cl_\omega(o(G))$.

THEOREM 3.2. Each one of the identity maps $i: Y^* \rightarrow Y^*$, and $i: Y^* \rightarrow Y^*$ is $\theta$-continuous.

THEOREM 3.3. The spaces $Y^*$, $Y^*$, $Y^*$, and $Y^*$ are $\theta$-homeomorphic. Moreover, $Y^*$, $Y^*$, and $Y^*$ are $\theta$-equivalent extensions of $X$ with homeomorphic remainders.

COROLLARY 3.1. If $(Y, \tau)$ is an extension of a space $(X, \tau')$, then the spaces $Y^*$, $Y^*$, $Y^*$, and $Y^*$ are homeomorphic in pairs. Moreover, the spaces $Y^*$, $Y^*$, and $Y^*$ are equivalent extensions of $X$.

REMARKS 3.1. (a) If $P$ is any property of topological spaces which is preserved under $\theta$-continuous surjections, and if $(Y, \tau)$ is a $P$-extension of $(X, \tau')$, then $Y$, $Y^*$, $Y^*$, and $Y^*$ are also $P$-extensions of $X$.

(b) The extensions $Y^*$, $Y^*$, $Y^*$, and $Y^*$ introduced above are, in general, all distinct from $Y$, $Y^*$, and $Y^*$. It would be interesting to find a characterization of spaces $Y$ for which $Y^* = Y$. A space $Z$ is called $H$-closed if it is closed in every Hausdorff space in which it is embedded [see 11 for more details]. The Katetov (respectively, Fomin) extension of a space $(X, \tau')$ is the space $\kappa X$ (respectively, $\sigma X$) whose underlying set is the set $X \cup \{p: p$ is a free open ultrafilter on $X\}$, and whose topology has for an open base the family $\tau' \cup \{U \cup \{p\}: U \in p, p \in \kappa X \setminus X\}$ (respectively, the family $\{o_x(U): U \in \tau'\}$). The spaces $\kappa X$, and $\sigma X$ are $H$-closed extensions of $X$ such that $(\sigma X)^* = \kappa X$, and $(\kappa X)^* = \sigma X$ [3, 6, 11]. In general $(\kappa X)^* \neq (\sigma X)^* = \kappa X$, and $(\kappa X)^* \neq (\sigma X)^* = \sigma X$. Analogous remarks apply to the Banaschewski-Fomin-Shanin extension $\mu X$ [13] of a Hausdorff space $X$.

(c) A space $Z$ is called compact like, or nearly compact if every regular open cover of $Z$ is reducible to a finite subcover. A space $X$ has a compactlike extension if and only if $X_\omega$ is Tychonoff [14]. Compactlike extensions (=near compactifications) of Hausdorff almost completely regular spaces $X$ (whence, $X_\omega$ is Tychonoff) have been constructed in [2] via $\theta$-proximities. For a Hausdorff space $X$ whose semiregularization $X_\omega$ is Tychonoff, a maximal compactlike extension $BX$ of $X$, satisfying $(BX)_\omega = \beta X$, is constructed in [14]. If $(X, \tau')$ is any Hausdorff almost completely regular space, and if $(Y, \tau)$ is any near compactification of $(X, \tau')$, then so are $Y^*$, $Y^*$, $Y^*$, and $Y^*$.

(d) A space $Z$ is called almost real compact if every open ultrafilter on $Z$ with countable closed intersection property in $Z$ converges in $Z$ [4]. A space $Z$ is almost realcompact if and only if $Z_\omega$ is almost realcompact [12]. Almost realcompactifications of a Hausdorff space have been constructed (among others) in [7], and [12]. If $(X, \tau')$ is any Hausdorff space, and if $(Y, \tau)$ is any almost realcompactification of $(X, \tau')$, then so are $Y^*$, $Y^*$, $Y^*$, and $Y^*$.

(e) A Hausdorff space $Z$ is called extremally disconnected if for each open subset $U$ of $Z$, $cl_\omega(U)$ is open. A space $Z$ is extremally disconnected if and only if each dense subspace of $Z$ [respectively, if and only if $Z_\omega$] is extremally disconnected [see 11 for more details]. A Hausdorff space $Z$ is called $s$-closed if it is $H$-closed and extremally disconnected [8]. A Hausdorff space $Z$ is $s$-closed if and only if $Z_\omega$ is $s$-closed. It is shown in [8] that every extremally disconnected space $X$ admits an $s$-closed extension, viz.
moreover, an extension \( Y \) of \( X \) is s-closed if and only if \( X \) is \( C' \)-embedded in \( Y \). If \( (X, \tau') \) is any extremally disconnected Hausdorff space, and if \( (Y, \tau) \) is any s-closed extension of \( (X, \tau') \), then so are \( y', y'', y^*, \) and \( y^{**} \).

REFERENCES


