L₁ SPACES FAIL A CERTAIN APPROXIMATIVE PROPERTY

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ABSTRACT. In this paper the author studies some cases of Banach space that does not have the property \( P_1 \). He shows that if \( X = \ell_1 \) or \( L_1(\mu) \) for some non-purely atomic measure \( \mu \), then \( X \) does not have the property \( P_1 \). He also shows that if \( X = \ell_\infty \) or \( C(Q) \) for some infinite compact Hausdorff space \( Q \), then \( X^* \) does not have the property \( P_1 \).

KEY WORDS AND PHRASES: Property \( P_1 \), classical Banach spaces \( \ell_1 \), \( L_1(\mu) \ell_1^* \), compact width


1. INTRODUCTION

The Banach space \( X \) is said to have the property \( P_1 \), if for each \( \epsilon > 0 \) and each \( r > 0 \), there is \( \delta > 0 \), such that for each \( x \) and \( y \) in \( X \), there is \( z \in B(x, \epsilon) \) satisfying that for each \( \theta \) with \( 0 < \theta < \delta \)

\[ B(x, r + \theta) \cap B(y, r + \theta) \subseteq B(z, \epsilon) \]

where \( B(x, r) \) is the open ball of radius \( r \) and centered at \( x \), and \( \overline{B(x, r)} \) is its clouser.

The property \( P_1 \) plays an important role in approximation theory, and many authors used it. This property appears in approximation by compact operators, simultaneous approximation and other areas (see for example Roversi [1], Lau [2], Mach [3] and Kamal [4]). Mach [3] showed that if \( X \) is uniformly convex then it has the property \( P_1 \) [3], and that if \( X = C(Q) \), or \( X = B(Q) \) then \( X \) has the property \( P_1 \) [4]. Mach [4, page 259] asked if the space \( L_1(\mu) \) has the property \( P_1 \).

In this paper the author studies some cases of normed linear space \( X \), for which \( X \) does not have the property \( P_1 \). In section 2, it is shown that if \( X = \ell_1 \) then \( X \) does not have the property \( P_1 \), and in section 3, it is shown that if \( \mu \) is a non-purely atomic measure, then \( L_1(\mu) \) does not have the property \( P_1 \). These two results give a negative answer for the question of Mach [4]. In section 3, it will be shown also that if \( X = (\ell_\infty)^* \), or \( X = (C(Q))^* \), where \( Q \) is an infinite compact Hausdorff space, then \( X \) does not have the property \( P_1 \).

In this paper \( \ell_1 \) is the Banach space of all real sequences \( x = \{x_i\} \) satisfying that \( \sum |x_i| < \infty \), together with the norm \( \|x\| = \sum |x_i| \). Also \( \ell_1^* \) is the Banach space of all real \( n \)-tuples \( x = (x_1, x_2, \ldots, x_n) \) together with the norm \( \|x\| = \sum_{i=1}^{n} |x_i| \).

2. \( \ell_1 \) DOES NOT HAVE THE PROPERTY \( P_1 \)

The proof of the fact that \( \ell_1 \) does not have the property \( P_1 \) depends on the behavior of the property \( P_1 \) in \( \ell_1^* \). In Lemma 2.3, it will be shown that if \( \epsilon > 0 \) is fixed, and \( \epsilon_n \) corresponds to \( \epsilon \) for \( X = \ell_1^* \) in Lemma 2.1, then \( \delta_n \to 0 \) when \( n \to \infty \), so using the fact that \( \ell_1^* \) is a norm-one-complemented subspace of \( \ell_1 \), it will be shown in Theorem 2.4, that \( \ell_1 \) does not have the property \( P_1 \).
**Lemma 2.1.** If the Banach space $X$ has the property $P_1$ then for each $\epsilon > 0$, there is $\delta > 0$ such that for each $y \in X$, there is $z \in B(0, \epsilon)$ such that if $0 < \theta < \delta$ then

$$B(0, 1 + \delta + \theta) \cap B(y, 1 + \theta) \subseteq B(z, 1 + \theta).$$

**Proof.** Let $r = 1$ and let $\epsilon > 0$ be given. By the definition of the property $P_1$, there is $\delta > 0$ such that for each $x$ and $y$ in $X$, there is $z \in B(x, \epsilon)$ satisfying the following: for each $\theta'$ such that $0 < \theta' < \delta'$

$$B(x, 1 + \delta') \cap B(y, 1 + \theta) \subseteq B(z, 1 + \theta').$$

Let $x = 0$ and $\delta = 1/2 \delta'$, then for all $\theta$ satisfying $0 < \theta < \delta$,

$$B(0, 1 + \delta + \theta) \cap B(y, 1 + \theta) \subseteq B(0, 1 + \theta + \delta) \cap B(y, 1 + \theta) \subseteq B(z, 1 + \theta).$$

**Lemma 2.2.** Let $n \geq 3$ be a positive integer, let $\delta > 0$ be given and let $(z_1, \ldots, z_n)$ be an $n$-tuple of real numbers

If $\sum_{i=1}^{n} z_i \geq \delta$, and for each $i \leq n - 1$

$$z_1 + \ldots + z_{i-1} - z_i + z_{i+1} + \ldots + z_n \leq -\delta,$$

then $\sum_{i=1}^{n} |z_i| \geq (2n - 3)\delta$.

**Proof.** For each $i = 1, 2, \ldots, n$, let $y_i = z_1 + \ldots + z_{i-1} - z_i + z_{i+1} + \ldots + z_n$, then

$$y_n = (n - 2) \sum_{i=1}^{n} z_i,$$

Thus

Therefore

$$\sum_{i=1}^{n} |z_i| \geq |y_n| \geq (n - 2)\delta + (n - 1)\delta = (2n - 3)\delta.$$

**Lemma 2.3.** Let $n \geq 3$ be a positive integer and let $\delta > 0$ be a given real number such that $(2n - 5)\delta \leq 1$. Then the element $x_n = (\delta, \delta, \ldots, \delta, -(n - 2)\delta)$ in $\ell^n_1$ satisfies the following conditions

1. $\forall \theta$ such that $\theta > 0$

$$B(0, 1 + \delta + \theta) \cap B(x_n, 1 + \theta) \neq \varnothing.$$

2. If $z \in \ell^n_1$ and for each $\theta$ with $0 < \theta < \delta$

$$B(0, 1 + \delta + \theta) \cap B(x_n, 1 + \theta) \subseteq B(z, 1 + \theta),$$

then $||z|| \geq (2n - 3)\delta$

**Proof.** Let $\{e_i\}_{i=1}^{n}$ be the standard basis in $\ell^n_1$, that is $e_i = (x_1, x_2, \ldots, x_n)$, where $x_j = 1$ if $i = j$ and $x_j = 0$ if $i \neq j$ and let

$$x_n = \delta \sum_{i=1}^{n-1} e_i - (n - 2)\delta e_n = (\delta, \delta, \ldots, \delta, -(n - 2)\delta) \in \ell^n_1.$$

Then $||x_n|| = (n - 1)\delta + (n - 2)\delta = (2n - 3)\delta \leq 1 + 2\delta \leq 2 + \delta$, therefore for each $\theta$ such that $0 < \theta < \delta$

$$B(0, 1 + \delta + \theta) \cap B(x_n, 1 + \theta) \neq \varnothing.$$

Assume that $z = (z_1, z_2, \ldots, z_n) \in \ell^n_1$ is such that for each $\theta$ with $0 < \theta < \delta$

$$B(0, 1 + \delta + \theta) \cap B(x_n, 1 + \theta) \subseteq B(z, 1 + \theta).$$
It will be shown that:

1. \( \sum_{i=1}^{n} z_i \geq \delta \), and
2. for each \( i \leq n - 1 \)
   \[ z_1 + \ldots + z_{i-1} - z_i + z_{i+1} + \ldots + z_n \leq -\delta. \]

If these are true then by Lemma 2.2, \( \|z\| = \sum_{i=1}^{n} |z_i| \geq (2n - 3)\delta \)

1. Assume that \( z_1 + \ldots + z_n < \delta \). Let
   \[
y = \delta \sum_{i=1}^{n-2} e'_i + \frac{1+\delta}{2} e'_n + \left[ \frac{1-(2n-5)}{2} \right] e'_n
   = \left( \delta, \ldots, \delta, \frac{1+\delta}{2}, \frac{1-(2n-5)}{2} \right) \in \ell_1^n.
   \]
   Then
   \[ \|y\| = (n-2)\delta + \frac{1+\delta}{2} + \frac{1-(2n-5)\delta}{2} = 1 + \delta. \]

On the other hand
\[ \|y - x_n\| = \left| \frac{1+\delta}{2} - \delta \right| + \left| \frac{1-(2n-5)\delta}{2} + (n-2)\delta \right| = 1. \]
Thus, for each \( \theta \) such that \( 0 < \theta < \delta \),
\[ y \in B(0, 1+\delta + \theta) \cap B(x_n, 1+\theta). \]

But
\[ \|y - z\| = \sum_{i=1}^{n} |y_i - z_i| \geq \left| \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i \right| = \left| 1 + \delta - \sum_{i=1}^{n} z_i \right|
   = 1 + \left( \delta - \sum_{i=1}^{n} z_i \right)
   > 1, \]
so for any \( \theta < \left( \delta - \sum_{i=1}^{n} z_i \right), \ y \notin B(z, 1+\theta). \)

2. Assume that for a certain \( i_0 \leq n - 1 \)
   \[ z_1 + \ldots + z_{i_0-1} - z_{i_0} + z_{i_0+1} + \ldots + z_n > -\delta. \]
Let
\[ y = \left( \frac{1+\delta}{2} \right) e'_{i_0} - \left( \frac{1+\delta}{2} \right) e'_n = \left( 0, 0, \ldots, 0, \frac{1+\delta}{2}, 0, \ldots, 0, -\left( \frac{1+\delta}{2} \right) \right) \in \ell_1^n. \]

\( i_0 \)-th term
Then
\[ \|y\| = 1 + \delta, \]
and
Thus, for each $\theta$ such that $0 < \theta < \delta$, $y \in B(0, 1 + \delta + \theta) \cap B(x_{n}, 1 + \theta)$. But

$$
\|y - x_{n}\| = (n - 2)\delta + \frac{1 + \delta}{2} - \delta + \frac{1 + \delta}{2} + (n - 2)\delta \\
= (n - 2)\delta + \frac{1 - \delta}{2} + \frac{1 - (2n - 5)\delta}{2}
$$

Thus, for each $\theta$ such that $0 < \theta < \delta$, $y \in B(0, 1 + \delta + \theta) \cap B(x_{n}, 1 + \theta)$. But

$$
\|y - z\| = \sum_{i=1}^{n} |y_{i} - z_{i}|
$$

Thus, for some $\theta > 0$, $y \notin B(z, 1 + \theta)$.

**THEOREM 2.4.** $\ell_{1}$ does not have the property $P_{1}$

**PROOF.** It will be shown that for each $\delta > 0$, there is $x_{\delta} \in \ell_{1}$, such that if $z \in \ell_{1}$ and for all $\theta$ with $0 < \theta < \delta$ it is true that $B(0, 1 + \delta + \theta) \cap B(x_{\delta}, 1 + \theta) \subseteq B(z, 1 + \theta)$, then $\|z\| > \frac{1}{\delta}$. Let $\{e_{i}\}_{i=1}^{\infty}$ be the standard basis in $\ell_{1}$, and let $\delta > 0$ be given. If $\delta > 1$ then for each $\theta > 0$

$$
B(0, 1 + \theta) \cap B(x_{1}, 1 + \theta) \subseteq B(0, 1 + \delta + \theta) \cap B(x_{1}, 1 + \delta).
$$

Thus one can take $x_{1}$ to be $x_{\delta}$. So without loss of generality one may assume that $\delta \leq 1$.

Let $n \geq 3$ be a positive integer satisfying $(2n - 5)\delta \leq 1$ and $(2n - 3)\delta > \frac{1}{\delta}$, and let $x_{n}$ be as in Lemma 2.3. Define

$$
x_{\delta} = \delta \sum_{i=1}^{n-1} e_{i} - (n - 2)\delta e_{n} = (\delta, \delta, \ldots, \delta, -(n - 2)\delta, 0, 0, \ldots) \in \ell_{1}.
$$

Then $\|x_{\delta}\| = \|x_{n}\| \leq 2 + \delta$, thus

$$
B(0, 1 + \delta + \theta) \cap B(x_{\delta}, 1 + \theta) \neq \phi \quad \text{for} \quad 0 < \theta < \delta.
$$

Let $P_{n} : \ell_{1} \to \ell_{1}^{n}$ be the mapping defined by $P_{n}(\{x_{i}\}_{i=1}^{\infty}) = \{x_{i}\}_{i=1}^{n}$. By the construction of $x_{\delta}$ its image under $P_{n}$ is the element $x_{n}$. Assume that for some $z \in \ell_{1}$

$$
B(0, 1 + \delta + \theta) \cap B(x_{\delta}, 1 + \theta) \subseteq B(z, 1 + \theta), \quad 0 < \theta < \delta,
$$

then in $\ell_{1}^{n}$

$$
B(0, 1 + \delta + \theta) \cap B(x_{n}, 1 + \theta) \subseteq B(P_{n}(z), 1 + \theta), \quad 0 < \theta < \delta.
$$

Thus by Lemma 2.3 $\|P_{n}(z)\| \geq (2n - 3)\delta > \frac{1}{\delta}$. Therefore

$$
\|z\| \geq \|P_{n}(z)\| > \frac{1}{\delta}.
$$

**3. OTHER SPACES THAT DO NOT HAVE THE PROPERTY $P_{1}$**

The subspace $Y$ of $X$ is called a norm-one-complemented subspace of $X$ if there is a linear projection $P : X \to Y$ satisfying that $\|P\| = 1$. If $A$ is a subset of $X$, and $x \in X$ then

$$
d(x, A) = \inf\{\|x - y\|; y \in A\},
$$

and if $B$ is another subset of $X$, then the deviation of $A$ from $B$ is defined by

$$
\delta(A, B) = \sup\{d(x, B); x \in A\}.
$$
The compact width of $A$ in $X$ is defined by
$$\alpha(A, X) = \inf \{ \delta(A, K); K \text{ is a compact subset of } X \}.$$ 

The compact width is said to be attained if there is a compact subset $K$ of $X$ satisfying that $\alpha(A, X) = \delta(A, K)$.

In this section it will be shown that if $X = (C(Q))^*$, where $Q$ is an infinite compact Hausdorff space, $X = (\ell_\infty)^*$, or $X = L_1(\mu)$ where $\mu$ is non-purely atomic measure, then $X$ does not have the property $P_1$.

The proof of the following proposition is elementary

**PROPOSITION 3.1.** Let $X$ be a Banach space that has the property $P_1$, and let $Y$ be a closed subspace of $X$. If $Y$ is a norm-one-complemented subspace of $X$, then $Y$ has the property $P_1$.

**COROLLARY 3.2.** If $\mu$ is non-purely atomic measure then $L_1(\mu)$ does not have the property $P_1$.

**PROOF.** By Feder [5, Theorem 2], $L_1[0, 1]$ has a subset $A$ for which the compact width $\alpha(A, L_1[0, 1])$ is not attained, thus by Kamal [6, Theorem 4.3] $L_1[0, 1]$ does not have the property $P_1$, but by Lacy [7, sec 8], $L_1[0, 1]$ is a norm-one-complemented subspace of $L_1(\mu)$, therefore by Proposition 3.1, $L_1(\mu)$ does not have the property $P_1$.

**NOTE 3.3.** Theorem 2.4 together with Corollary 3.2 give a negative answer to the question of Mach [4, page 259].

**COROLLARY 3.4.** If $X = \ell_\infty$ or $X = C(Q)$ for some compact infinite Hausdorff space $Q$, then $X^*$ does not have the property $P_1$.

**PROOF.** If $X = \ell_\infty$, then $\ell_1$ is a norm-one-complemented subspace of $X^*$, and if $X = C(Q)$ then by Kamal [8, Lemma 3.2], $\ell_1$ is a norm-one-complemented subspace of $X^*$, in both cases one concludes by Proposition 3.1 that $X^*$ does not have the property $P_1$.

**REFERENCES**


