GENERALIZED FRACTIONAL CALCULUS TO A SUBCLASS OF ANALYTIC FUNCTIONS FOR OPERATORS ON HILBERT SPACE

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ABSTRACT. In this paper, we investigate some generalized results of applications of fractional integral and derivative operators to a subclass of analytic functions for operators on Hilbert space.

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1. INTRODUCTION AND DEFINITIONS

Let \( A \) denote the class of functions of the form:

\[
f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \quad (a_1 := 1),
\]

which are analytic in the open unit disk

\[ U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}. \]

Also let \( S \) denote the class of all functions in \( A \) which are univalent in the unit disk \( U \).

Let \( S_0(\alpha, \beta, \gamma, p) \) denote the class of functions

\[
f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_{n+p} \geq 0),
\]

which are analytic and \( p \)-valent in \( U \) and satisfy the condition

\[
\left| \frac{zf'(z)}{f(z)} - p \right| < \beta \left| \alpha \frac{zf'(z)}{f(z)} + (p - \gamma) \right|
\]

for \( 0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \gamma < p, p \in \mathbb{N} \) and \( z \in U \). See Lee et al [1] for further information on them. It is easily found that \( S_0(\alpha, \beta, \gamma, p) \subset A \) when \( p = 1 \).

Let \( a, b, \) and \( c \) be complex numbers with \( c \neq 0, -1, -2, \cdots \). Then the Gaussian hypergeometric function \( \,\!_2F_1(z) \) is defined by

\[
_2F_1(z) \equiv _2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,
\]

where \( (\lambda)_n \) is the Pochhammer symbol defined, in terms of the Gamma function, by

\[
(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N} := \{1, 2, \cdots \}) \end{cases}.
\]
Let $A$ be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. For a complex valued function $f$ analytic on a domain $E$ of the complex plane containing the spectrum $\sigma(A)$ of $A$ we denote $f(A)$ as Riesz-Dunford integral [2, p. 568], that is,

$$f(A) := \frac{1}{2\pi i} \int_C f(z)(zI - A)^{-1}dz,$$

where $I$ is the identity operator on $\mathcal{H}$ and $C$ is positively oriented simple closed rectifiable contour containing $\sigma(A)$.

Also $f(A)$ can be defined by the series $f(A) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} A^n$ which converges in the norm topology [3].

Xiaopei [4] defined $S_0(\alpha, \beta, \gamma, p; A)$ by the class of functions

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p}z^{n+p} \quad (a_{n+p} \geq 0),$$

which is analytic and $p$-valent in $\mathcal{U}$ and satisfies the condition,

$$\|Af'(A) - pf(A)\| < \beta\|\alpha Af'(A) + (p - \gamma)f(A)\|$$

for $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $0 \leq \gamma < p$, $p \in \mathbb{N}$ and all operators $A$ with $\|A\| < 1$ and $A \neq 0$ (denotes the zero operator on $\mathcal{H}$).

Let $A^*$ denote the conjugate operator of $A$.

DEFINITION 1 ([4]). The fractional integral for operator of order $a$ is defined by

$$D_{-}^{a}f(A) = \frac{1}{\Gamma(a)} \int_0^1 A^{a}f(tA)(1-t)^{a-1}dt,$$

where $a > 0$ and $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin.

DEFINITION 2 ([4]). The fractional derivative for operator of order $a$ is defined by

$$D_{+}^{a}f(A) = \frac{1}{\Gamma(1-a)} g'(A),$$

where $g(z) = \int_0^1 z^{1-a}f(tz)(1-t)^{-a}dt$ ($0 < a < 1$) and $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin.

Srivastava et al. [5] introduced a fractional integral operator $I_{0+}^{a,b,c}$ defined by (cf. [6])

$$I_{0+}^{a,b,c}f(z) = z^{-b} \Gamma(a) \int_0^1 (1-t)^{a-1} 2F_1(a+b, -c; a; 1-t)f(tz)dt$$

$$a > 0; b, c \in \mathbb{R}; f(z) \in \mathcal{A}.$$  

and Owa et al. [7] studied the fractional operator $J_{0+}^{a,b,c}$ defined by (see also Kim et al. [8])

$$J_{0+}^{a,b,c}f(z) = \frac{\Gamma(2-b)\Gamma(2+a+c)}{\Gamma(2-b+c)} z^b I_{0+}^{a,b,c}f(z) \quad (f \in \mathcal{A}).$$

The fractional derivative operator $D_{0+}^{a,b,c}$ is defined by (cf. [9])

$$D_{0+}^{a,b,c}f(z) = \frac{d}{dz} \left( z^{-b} \Gamma(1-a) \int_0^1 (1-t)^{-a} 2F_1(b-a+1, -c; 1-a; 1-t)f(tz)dt \right)$$

$$0 \leq a < 1; b, c \in \mathbb{R}; f(z) \in \mathcal{A}.$$  

And we define $D_{0+}^{n+a,b,c}$ by
For all invertible operator $A$, we introduce the following definition:

**DEFINITION 3.** The fractional integral operator for operator $A^{\alpha,b,c}$ is defined by

$$I_{0,A}^{\alpha,b,c} f(A) = \frac{1}{\Gamma(a)} \int_0^1 A^{-b} \, T_2 F_1(a+b-c; a; 1-t) f(tA)(1-t)^{a-1} dt,$$

where $a > 0$ and $b, c \in \mathbb{R}$.

The fractional derivative operator for operator $A^{\alpha,b,c}$ is defined by

$$D_{0,A}^{\alpha,b,c} f(A) = \frac{1}{\Gamma(1-a)} g'(A),$$

where

$$g(z) = \int_0^1 z^{-b} \, T_2 F_1(b-a+1, -c; 1-a; 1-t) f(tz)(1-t)^{-a} dt,$$

$0 < a < 1$ and $b, c \in \mathbb{R}$. In both (1.14) and (1.15) $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin with the order

$$f(z) = O(|z|^\epsilon), \quad z \to 0,$$

where $\epsilon > \max\{0, b - c\} - 1$ and the multiplicity of $(1-t)^{a-1}$ is in (1.14) and that of $(1-t)^{-a}$ in (1.15) removed by requiring $\log(1-t)$ to be real when $1 - t > 0$.

We note that

$$A^{a,b,c} f(A) = D_{0,A}^{a} f(A)$$

The object of this paper is to prove the distortion theorems of fractional integral and derivative operators to $S_0(\alpha, \beta, \gamma, p; A)$.

2. **RESULTS**

**LEMMA 1** (Xiaopei [4, Theorem 2.1]. An analytic function $f(z)$ is in the class $S_0(\alpha, \beta, \gamma, p; A)$ for all proper contraction $A$ with $A \neq 0$ if and only if

$$\sum_{k=1}^{\infty} \{|k + \beta[p - \gamma + \alpha(k + p)]\} a_k + p \leq \beta(p - \gamma + \alpha p)$$

for $0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \gamma < p$, and $p \in \mathbb{N}$.

The result is sharp for the function

$$f(z) = z^p - \frac{\beta(p - \gamma + \alpha p)}{k + \beta[p - \gamma + \alpha(k + p)]} z^{k+p} \quad (k \geq 1).$$

**THEOREM 1.** Let $p > \max\{b - c - 1, b - 1, -1 - c - a\}$ and $a(p + 1) > b(a + c)$. If $f(z) \in S_0(\alpha, \beta, \gamma, p; A)$, then

$$\|I_{0,A}^{\alpha,b,c} f(A)\| \leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p+1-b)\Gamma(a+p+1+c)} \|A\|^{-b}$$

$$+ \frac{\beta(p - \gamma + \alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\{1 + \beta[p - \gamma + \alpha(p+1)]\}\Gamma(p+1-b)\Gamma(a+p+1+c)} \|A\|^{p+1-b}$$

and

$$\|D_{0,A}^{\alpha,b,c} f(A)\| \leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p+1-b)\Gamma(a+p+1+c)} \|A\|^{-b}$$

$$+ \frac{\beta(p - \gamma + \alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\{1 + \beta[p - \gamma + \alpha(p+1)]\}\Gamma(p+1-b)\Gamma(a+p+1+c)} \|A\|^{p+1-b}$$

and
\[ \|f(A)\| \geq \frac{\Gamma(p + 1 - b + c)\Gamma(p + 1)}{\Gamma(p + 1 - b)\Gamma(a + p + 1 + c)} \|A\|^{-b} \]

\[ - \frac{\beta(p - \gamma + \alpha)}{1 + \beta[p - \gamma + \alpha(p + 1)]} \|A\|^{p+1} \|A^{-b}\| \]  

(2.3)

for \( a > 0, b, c \in \mathbb{R} \) and all invertible operator \( A \) with \((A^\frac{1}{q})^* A^\frac{1}{q} = A\) \((q \in \mathbb{N})\), \(\|A\| < 1\) and \(r_{sp}(A) r_{sp}(A^{-1}) \leq 1\), where \(r_{sp}(A)\) is the radius of spectrum of \(A\).

**Proof.** Consider the function

\[ F(A) = \frac{\Gamma(p + 1 - b)\Gamma(a + p + 1 + c)}{\Gamma(p + 1 - b + c)\Gamma(p + 1)} A^b r_{0,a} f(A) \]

\[ = A^p - \sum_{k=1}^{\infty} B_{k+p} A^{k+p} \]

(2.4)

where

\[ B_{k+p} = \frac{\Gamma(k + p + 1 - b + c)\Gamma(p + 1 + k)\Gamma(p + 1 - b)\Gamma(a + p + 1 + c)}{\Gamma(k + p + 1 - b)\Gamma(a + k + p + 1 + c)\Gamma(p + 1)\Gamma(p + 1 - b + c)} \]

Hence, for convenience, we put

\[ \Phi(k) = \frac{\Gamma(k + p + 1 - b + c)\Gamma(p + 1 + k)\Gamma(p + 1 - b)\Gamma(a + p + 1 + c)}{\Gamma(k + p + 1 - b)\Gamma(a + k + p + 1 + c)\Gamma(p + 1)\Gamma(p + 1 - b + c)} a_{k+p} \]

(2.5)

Then, by the constraints of the hypotheses, we note that \(\Phi(k)\) is non-increasing for integers \( k \geq 1 \) and we have \( 0 < \Phi(k) < 1 \). So \( F(z) \in S_0(\alpha, \beta, \gamma, p; A) \). By Lemma 1, we get

\[ \{1 + \beta[p - \gamma + \alpha(p + 1)]\} \sum_{k=1}^{\infty} B_{k+p} \leq \sum_{k=1}^{\infty} \{k + \beta[p - \gamma + \alpha(k + p)]\} B_{k+p} \]

\[ \leq \sum_{k=1}^{\infty} \{k + \beta[p - \gamma + \alpha(k + p)]\} a_{k+p} \]

\[ \leq \beta(p - \gamma + \alpha p), \]

(2.6)

which gives

\[ \sum_{k=1}^{\infty} B_{k+p} \leq \frac{\beta(p - \gamma + \alpha p)}{1 + \beta[p - \gamma + \alpha(p + 1)]} . \]

Therefore, in a similar way with the proof of [4, Theorem 2.3, p. 305], we obtain

\[ \|r_{0,a} f(A)\| \geq \frac{\Gamma(p + 1 - b + c)\Gamma(p + 1)}{\Gamma(p + 1 - b)\Gamma(a + p + 1 + c)} \|A\|^{-b} \|A\|^{p} \]

\[ - \frac{\beta(p - \gamma + \alpha)}{1 + \beta[p - \gamma + \alpha(p + 1)]} \|A\|^{p+1} \|A^{-b}\| \]  

(2.7)

and

\[ \|r_{0,a} f(A)\| \leq \frac{\Gamma(p + 1 - b + c)\Gamma(p + 1)}{\Gamma(p + 1 - b)\Gamma(a + p + 1 + c)} \|A\|^{-b} \|A\|^{p} \]

\[ + \frac{\beta(p - \gamma + \alpha)}{1 + \beta[p - \gamma + \alpha(p + 1)]} \|A\|^{p+1} \|A^{-b}\| \]  

(2.8)

By equation (7) of [4, p.307],
\[ \|A^b\| = \|A\|^b \quad (b > 0). \]  
(2.9)

Since \( A^*A = AA^* \), \( \|A\| = r_{sp}(A) \). So

\[ 1 = \|AA^{-1}\| \leq \|A\| \|A^{-1}\| = r_{sp}(A)r_{sp}(A^{-1}) \leq 1. \]

Thus

\[ \|A^{-1}\| = \|A\|^{-1}. \]  
(2.10)

By (2.9) and (2.10),

\[ \|A^b\| = \|A\|^b \]  
(2.11)

for all real \( b \). Therefore from (2.7), (2.8) and (2.11) we have the desired estimates.

**THEOREM 2.** Let \( p > \max(b - c - 1, b, 2 - c + a), c + 1 < (p - b)(1 - a + p + c) \), and \( b(2 - a + c) \leq (1 - a)(1 + p) \). If \( f(z) \in \mathcal{S}_0(\alpha, \beta, \gamma, p; A) \), then

\[ \|D_{0, A}^{\alpha, \beta, c} f(A)\| \leq \frac{\Gamma(p + 1 - b + c)\Gamma(p + 1)}{\Gamma(p - b)\Gamma(2 - a + p + c)} \|A\|^{p-b-1} \]

\[ + \frac{\beta(p + 1)(p - \gamma + \alpha p)\Gamma(p + 1 - b + c)\Gamma(p + 1)}{\{1 + \beta[p - \gamma + \alpha(p + 1)]\}\Gamma(p - b)\Gamma(2 - a + p + c)} \|A\|^p \]  
(2.12)

and

\[ \|D_{0, A}^{\alpha, \beta, c} f(A)\| \geq \frac{\Gamma(p + 1 - b + c)\Gamma(p + 1)}{\Gamma(p - b)\Gamma(2 - a + p + c)} \|A\|^{p-b-1} \]

\[ - \frac{\beta(p + 1)(p - \gamma + \alpha p)\Gamma(p + 1 - b + c)\Gamma(p + 1)}{\{1 + \beta[p - \gamma + \alpha(p + 1)]\}\Gamma(p - b)\Gamma(2 - a + p + c)} \|A\|^p \]  
(2.13)

for \( 0 < \alpha < 1, b, c \in \mathbb{R} \) and all invertible operator \( A \) with \( (A^\frac{1}{2})^*A^\frac{1}{2} = A^\frac{1}{2}(A^\frac{1}{2})^* \) \((q \in \mathbb{N})\), \( \|A\| < 1 \) and \( r_{sp}(A)r_{sp}(A^{-1}) \leq 1 \), where \( r_{sp}(A) \) is the radius of spectrum of \( A \).

**PROOF.** Consider the function

\[ G(A) = \frac{\Gamma(p - b)\Gamma(2 - a + p + c)}{\Gamma(p + 1 - b + c)\Gamma(p + 1)} A^{p+1}D_{0, A}^{\alpha, \beta, c} f(A) \]

\[ = A^p - \sum_{k=1}^{\infty} \frac{\Gamma(k + p + 1 - b + c)\Gamma(p + 1 + k)\Gamma(p - b)\Gamma(2 - a + p + c)}{\Gamma(k + p - b)\Gamma(2 - a + k + p + c)\Gamma(p + 1)\Gamma(p + 1 - b + c)} a_{k+p}A^{k+p} \]

\[ = A^p - \sum_{k=1}^{\infty} C_{k+p}A^{k+p}, \]  
(2.14)

where

\[ C_{k+p} = \frac{\Gamma(k + p + 1 - b + c)\Gamma(p + 1 + k)\Gamma(p - b)\Gamma(2 - a + p + c)}{\Gamma(k + p - b)\Gamma(2 - a + k + p + c)\Gamma(p + 1)\Gamma(p + 1 - b + c)} a_{k+p}. \]

Hence, for convenience, we put

\[ \Psi(k) = \frac{\Gamma(k + p + 1 - b + c)\Gamma(p + 1 + k)\Gamma(p - b)\Gamma(2 - a + p + c)}{\Gamma(k + p - b)\Gamma(2 - a + k + p + c)\Gamma(p + 1)\Gamma(p + 1 - b + c)} \quad (k \in \mathbb{N}). \]  
(2.15)

Then, by the constraints of the hypotheses, we note that \( \Psi(k) \) is non-increasing for integers \( k \geq 1 \) and we have \( 0 < \Psi(k) < 1 \), i.e.,

\[ 0 < \frac{\Gamma(k + p + 1 - b + c)\Gamma(p + 1 + k)\Gamma(p - b)\Gamma(2 - a + p + c)}{\Gamma(k + p - b)\Gamma(2 - a + k + p + c)\Gamma(p + 1)\Gamma(p + 1 - b + c)} < k + p. \]

Also, by the relation

\[ \frac{k + p}{p + 1} \{1 + \beta[p - \gamma + \alpha(p + 1)]\} \leq k + \beta[p - \gamma + \alpha(p + k)] \quad (k \geq 1), \]  
(2.16)

we get
\[ \sum_{k=1}^{\infty} \frac{k+p}{p+1} \left\{ 1 + \beta[p - \gamma + \alpha(p+1)] \right\} \Psi(k) a_{k+p} \leq \sum_{k=1}^{\infty} \left\{ \frac{k+\beta[p - \gamma + \alpha(k+p)]}{1 + \beta[p - \gamma + \alpha(p+1)]} \right\} \Psi(k) a_{k+p} \]
\[ \leq \beta(p - \gamma + \alpha p), \] (2.17)

that is,
\[ \sum_{k=1}^{\infty} (k+p) \Psi(k) a_{k+p} \leq \frac{\beta(p+1)(p - \gamma + \alpha p)}{1 + \beta[p - \gamma + \alpha(p+1)]} . \]

Therefore, in the same way with the proof of Theorem 1, we obtain
\[ \| D_{0,A}^{a,b,c} f(A) \| \leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^p-b-1 \]
\[ + \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^p-b \sum_{k=1}^{\infty} (k+p) \Psi(k) a_{k+p} \]
\[ \leq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^p-b-1 \]
\[ + \frac{\beta(p+1)(p - \gamma + \alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^p-b \] (2.18)

and
\[ \| D_{0,A}^{a,b,c} f(A) \| \geq \frac{\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^p-b-1 \]
\[ - \frac{\beta(p+1)(p - \gamma + \alpha p)\Gamma(p+1-b+c)\Gamma(p+1)}{\Gamma(p-b)\Gamma(2-a+p+c)} \| A \|^p-b \] (2.19)

REMARK. (i) By the proof of Theorem 1, if we put
\[ F(z) = z^{a,b,c} f(z) := \frac{\Gamma(p+1-b+c)\Gamma(a+p+1+c)}{\Gamma(p+1-b-c)\Gamma(p+1)} z^b \Delta_{0,z}^{a,b,c} f(z), \] (2.20)

then we know that \( \Delta_{0,z}^{a,b,c} \) is a fractional linear operator from \( \Delta_0(\alpha, \beta, \gamma, p) \) to itself.

(ii) From (1.17) it is easy to see that Theorem 1 and Theorem 2 are generalizations of [4, Theorem 3.1 and Theorem 3.2].

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