ON A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS NEAR CRITICAL GROWTH

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ABSTRACT. We use Minimax Methods and explore compact embeddings in the context of Orlicz and Orlicz-Sobolev spaces to get existence of weak solutions on a class of semilinear elliptic equations with nonlinearities near critical growth. We consider both biharmonic equations with Navier boundary conditions and Laplacian equations with Dirichlet boundary conditions.

KEY WORDS AND PHRASES: Elliptic Equations, Variational Methods, Orlicz Spaces.

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1. INTRODUCTION

Our concern in this paper is on finding weak solutions for the problem

$$(-1)^m \Delta^m u = f(x, u) \text{ in } \Omega, \quad B_m(u) = 0 \text{ on } \partial\Omega \quad (1.1)$$

where Δ^m is the elliptic operator

$$\Delta^m \equiv \sum_{i=1}^N \frac{\partial^{2m}}{\partial x_i^{2m}} + (m-1) \sum_{\substack{i,j=1, \\ i \neq j}}^N \frac{\partial^{2m}}{\partial x_i^m \partial x_j^m} \quad m = 1, 2,$$

$f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and the boundary operator B_m is given by

$$B_m(u) = (u, (m-1)\Delta u),$$

that is, $B_m(u) = 0$ means either the Dirichlet or the Navier boundary conditions according to $m = 1$ or $m = 2$.

By a weak solution of (1.1) we mean an element $u \in H_m \equiv H_0^1(\Omega) \cap H^m(\Omega)$ satisfying

$$\langle u, v \rangle_m = \int_{\Omega} f(x, u)v, \quad v \in H_m$$

with $\Delta u = 0$ on $\partial\Omega$ when $m = 2$, where

$$\langle u, v \rangle_m \equiv (m-1) \int_{\Omega} \Delta u \Delta v + (2-m) \int_{\Omega} \nabla u \nabla v \quad u, v \in H_m.$$

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By the way $\langle \cdot, \cdot \rangle_m$ is an inner product in H_m , we denote by $\|\cdot\|_m$ its corresponding norm and we remark that H_m is a Hilbert space.

Now let $a : [0, \infty) \rightarrow \mathbb{R}$ be a right continuous, nondecreasing function satisfying the following conditions

$$a(0) = 0, \quad a(t) > 0 \text{ for } t > 0, \quad a(t) \rightarrow \infty \text{ as } t \rightarrow \infty \quad (1.2)$$

and let

$$A(t) = \int_0^t a(|s|)ds \quad \text{and} \quad p^* = \frac{2N}{(N-2m)}.$$

We shall assume that both

$$|f(x, t)| \leq C_1 + C_2 a(|t|), \quad (x, t) \in \Omega \times \mathbb{R} \quad (1.3)$$

for some $C_1 \geq 0, C_2 > 0$ and

$$A(t) = o(t^{p^*}) \text{ as } t \rightarrow \infty. \quad (1.4)$$

Now consider the functional

$$I_m(u) = \frac{1}{2} \|u\|_m^2 - \int_{\Omega} F(x, u) dx, \quad u \in H_m$$

where $F(x, t) = \int_0^t f(x, s) ds$. It follows under conditions (1.2)(1.3)(1.4) and condition (1.5) below that $I_m \in C^1(H_m, \mathbb{R})$ and its derivative is given by

$$\langle I'_m(u), v \rangle = \langle u, v \rangle_m - \int_{\Omega} f(x, u) v \quad u, v \in H_m.$$

We shall look for weak solutions of (1.1) by finding critical points of I_m . Our main result is the following.

THEOREM 1. Assume (1.2)(1.3)(1.4). Assume in addition that

$$a(|t|) \leq |t|^{(p^*-1)} \quad t \in \mathbb{R}, \quad (1.5)$$

$$f(x, t) = o(t) \quad t \rightarrow 0, \quad \text{uniformly } x \in \Omega \quad (1.6)$$

$$0 < \theta F(x, t) \leq t f(x, t) \quad \text{a.e. } x \in \Omega \quad |t| \geq M \quad (1.7)$$

for some $M > 0, \theta > 2$.

Then (1.1) has a non zero weak solution.

Our Theorem improves results by Rabinowitz [15], Gu [7], deFigueiredo, Clement & Mitidieri [3] in the sense that we allow less restrictive growth on $f(x, t)$. It is also related to some results in Brézis & Nirenberg [14], Pucci & Serrin [12], van der Vorst [13].

We employ the Ambrosetti & Rabinowitz Mountain Pass Theorem as in some of the above mentioned papers and the main point here is the use of Orlicz and Orlicz-Sobolev spaces to overcome compactness difficulties.

2. PRELIMINARIES

We shall apply the following variant of the Ambrosetti & Rabinowitz [2] Mountain Pass Theorem (see Mawhin & Willem [6]).

THEOREM 2. Let X be a Banach space and let $I \in C^1(X, \mathbb{R})$ with $I(0) = 0$. Assume in addition that

$$I(u) \geq r \quad \text{when} \quad \|u\| = \rho, \quad \text{for some } r, \rho > 0 \quad (2.1)$$

$$I(e) \leq 0, \quad \text{for some } e \in X \quad \text{with} \quad \|e\| > \rho. \quad (2.2)$$

Then there is a sequence $u_n \in X$ such that

$$I(u_n) \rightarrow c \quad \text{and} \quad I'(u_n) \rightarrow 0$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad c \geq r$$

and

$$\Gamma = \{\gamma \in C([0, 1], X) \mid \gamma(0) = 0, \gamma(1) = e\}.$$

We shall apply theorem 2 with $I = I_m$ and $X = H_m$. The two lemmas below are crucial in applying theorem 2 to prove theorem 1.

LEMMA 3. (The Mountain Pass Geometry) Assume (1.2)-(1.7). Then (2.1)-(2.2) hold true.

We remark that by lemma 3 there is a sequence $u_n \in H_m$ such that

$$I_m(u_n) \rightarrow c \quad \text{and} \quad I'_m(u_n) \rightarrow 0.$$

Such a sequence is called a $(PS)_c$ sequence.

We are going to show, (see lemma 5 below), that u_n has a convergent subsequence. The proof of lemma 5 uses a crucial compactness type result (see lemma 4 below).

Prior to stating lemma 4 we shall recall some notations and basic results on Orlicz and Orlicz-Sobolev spaces. We refer the reader to Krasnosels'kii & Rutickii [5], Gossez [4], Adams [1] for an accounting on the subject. In this regard a function A satisfying the set of conditions:

$$A \text{ is convex, even, continuous} \quad (2.3)$$

$$A(t) = 0 \quad \text{iff} \quad t = 0 \quad (2.4)$$

$$\frac{A(t)}{t} \rightarrow \begin{cases} 0 & \text{when } t \rightarrow 0 \\ \infty & \text{when } t \rightarrow \infty \end{cases} \quad (2.5)$$

is referred to in the literature on Orlicz Spaces as an N-function. An Orlicz space is defined by

$$L_A(\Omega) \equiv \{u : \Omega \rightarrow \mathbb{R} \mid u \text{ is measurable and } \int_{\Omega} A(l|u|) < \infty \text{ for some } l > 0\}$$

and the norm given by

$$|u|_A \equiv \inf_{\alpha \in \mathbb{R}} \left\{ \alpha > 0 \mid \int_{\Omega} A\left(\frac{|u|}{\alpha}\right) \leq 1 \right\}$$

turns it into a (not necessarily reflexive) Banach space and as a matter of fact $L_A(\Omega) \rightarrow L^1(\Omega)$.

Corresponding to A there is an N-function labeled $\bar{A}$ called the conjugate function of A which satisfies the so called Young's inequality

$$st \leq A(t) + \overline{A}(s)$$

and in addition

$$ta(t) = A(t) + \overline{A}(a(t))$$

where

$$A(t) = \int_0^t a(|s|)ds$$

and a satisfies (1.2).

Moreover one also has a Hölder inequality namely

$$\int u \cdot v \leq 2|u|_{L_A}|v|_{L_{\overline{A}}}.$$

Now the Orlicz-Sobolev space is defined by

$$W^m L_A(\Omega) = \{u \in L_A(\Omega) \mid D^\alpha u \in L_A(\Omega), \quad |\alpha| \leq m\}$$

and the norm

$$\|u\| \equiv \left[\sum_{|\alpha| \leq m} |D^\alpha u|_{L_A}^2 \right]^{\frac{1}{2}}$$

turns it into a Banach space.

LEMMA 4. Assume (1.4). Then $H_m \hookrightarrow L_A(\Omega)$, $m = 1, 2$

LEMMA 5. Assume (1.2) – (1.7). Then the sequence u_n has a convergent subsequence.

3. PROOFS.

PROOF OF LEMMA 3.

At first given $\epsilon > 0$ there is by (1.6) some $\delta > 0$ such that

$$\frac{f(x, t)}{t} \leq \epsilon, \quad |t| < \delta \quad \text{a.e. } x \in \Omega$$

so that

$$F(x, t) \leq \frac{\epsilon}{2}t^2, \quad |t| < \delta \quad \text{a.e. } x \in \Omega.$$

On the other hand from (1.3), (1.5) we have

$$|f(x, t)| \leq C_1 + C_2|t|^{(p^*-1)}$$

so that

$$F(x, t) \leq C_1|t| + \frac{C_2}{p^*}|t|^{p^*}, \quad \text{a.e. } x \in \Omega, \quad t \in \mathbb{R}.$$

Hence

$$F(x, t) \leq \frac{\epsilon}{2}t^2 + C_\delta|t|^{p^*}, \quad \text{a.e. } x \in \Omega, \quad t \in \mathbb{R}. \quad (3.1)$$

Now observing that

$$(m-1) \int_\Omega |\Delta u|^2 + (2-m) \int |\nabla u|^2 \geq \lambda_{1m} \int_\Omega u^2$$

where λ_{1m} is the first eigenvalue of

$$\begin{cases} (-1)^m \Delta^m u = \lambda u & \text{in } \Omega \\ B_m(u) = 0 & \text{on } \partial\Omega \end{cases}$$

and using (3.1) we get

$$\int_{\Omega} F(x, u) \leq \frac{\epsilon}{2\lambda_{1m}} |u|_m^2 + C_{\delta} \int_{\Omega} |u|^{p^*}$$

CLAIM 1. $|u|_{L^{p^*}} \leq C \|u\|_m$, $u \in H_m$.

Using CLAIM 1, we get

$$\int_{\Omega} F(x, u) \leq \frac{\epsilon}{2\lambda_{1m}} \|u\|_m^2 + C \|u\|_m^{p^*}$$

so that

$$I_m(u) \geq \left(\frac{1}{2} - \frac{\epsilon}{2\lambda_{1m}}\right) \|u\|_m^2 - C \|u\|_m^{p^*}.$$

Therefore there are $\rho > 0$, $r > 0$ such that

$$I_m(u) \geq r, \quad \|u\|_m = \rho.$$

On the other hand using (1.7) it follows that

$$F(x, t) \geq C |t|^{\theta}, \quad |t| \geq M \quad \text{a.e. } x \in \Omega.$$

Now take $\phi \in C_0^{\infty}$, $\phi \geq 0$, $\phi \not\equiv 0$ and $\lambda > 0$. Then

$$I_m(\lambda\phi) = \frac{\lambda^2}{2} \|\phi\|_m^2 - \int_{\lambda\phi \leq M} F(x, \lambda\phi) - \int_{\lambda\phi > M} F(x, \lambda\phi)$$

Since

$$F(x, \lambda\phi) \geq -C_1 \lambda\phi - C_2 A(\lambda\phi)$$

we get

$$\begin{aligned} I_m(\lambda\phi) &\leq \frac{\lambda^2}{2} \|\phi\|_m^2 + \int_{\lambda\phi \leq M} (C_1 \lambda\phi + C_2 A(\lambda\phi)) - \int_{\lambda\phi > M} F(x, \lambda\phi) \\ &\leq \frac{\lambda^2}{2} \|\phi\|_m^2 + \int_{\lambda\phi \leq M} (C_1 M + C_2 A(M)) - \int_{\Omega} \phi^{\theta} \chi_{\phi > \frac{M}{\lambda}} \\ &\leq \frac{\lambda^2}{2} \|\phi\|_m^2 + C_M - \lambda^{\theta} \int_{\Omega} \phi^{\theta} \chi_{\phi > \frac{M}{\lambda}}. \end{aligned}$$

Now, by Lebesgue Theorem

$$\int_{\Omega} \phi^{\theta} \chi_{\phi > \frac{M}{\lambda}} \rightarrow \int_{\Omega} \phi^{\theta}.$$

Thus

$$I_m(\lambda\phi) \rightarrow -\infty \quad \text{as } \lambda \rightarrow \infty.$$

VERIFICATION OF CLAIM 1. If $m = 1$ CLAIM 1 holds by the Sobolev inequality. So let us assume $m = 2$. Letting

$$\|u\|_{2,2} \equiv \max_{|\alpha| \leq 2} |D^{\alpha} u|_{L^2},$$

it is an easy matter to check that the space H_2 endowed with $\|\cdot\|_{2,2}$ is complete. We claim that

$$\|u\|_2 \leq C \|u\|_{2,2}.$$

Indeed,

$$\|u\|_2^2 = \int_{\Omega} |\Delta u|^2 \leq C \left(\max_i \int_{\Omega} \left| \frac{\partial^2 u}{\partial x_i^2} \right|^2 \right) \leq C \left(\max_{|\alpha| \leq 2} |D^{\alpha} u|_{L^2}^2 \right) = C \|u\|_{2,2}^2$$

Hence we also have

$$\|u\|_{2,2} \leq C \|u\|_2 \quad (3.2)$$

and by Sobolev embedding we get $|u|_{L^{p^*}} \leq C \|u\|_2$, showing CLAIM 1 and thus proving lemma 3.

The proof of lemma 4 is a consequence of a general result due to Donaldson & Trudinger [9] (see also Adams [1, Theorem 8.40]). For the sake of completeness we recall that result in an Appendix. (see THEOREM A.1)

PROOF OF LEMMA 4.

Case $m = 2$. Applying the notations of theorem A.1 let

$$B_0^{-1}(t) = \sqrt{2}t^{\frac{1}{2}}, \quad t \geq 0$$

and

$$(B_k)^{-1}(t) \equiv \int_0^t \frac{(B_{k-1})^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau \quad t \geq 0, \quad k = 1, 2.$$

We claim that

$$\int_1^{\infty} \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau = \infty \quad \text{for } k = 0, 1. \quad (3.3)$$

and

$$\int_1^{\infty} \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty \quad \text{for some } k \geq 2. \quad (3.4)$$

By (3.3) and (3.4) J is defined and $2 \leq J \leq N$.

Indeed by computing we find that

$$B_1^{-1}(t) = \frac{\sqrt{2}(N-2)}{2N} t^{\frac{N-2}{2N}} \quad (3.5)$$

and

$$B_2^{-1}(t) = \frac{\sqrt{2}(N-2)}{2N} t^{\frac{N-4}{2N}}. \quad (3.6)$$

Now using (3.5) and (3.6) and computing again we get (3.3). Thus $J \geq 2$.

In order to show (3.4) it suffices to evaluate

$$\int_0^t \frac{(B_{N-1})^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau.$$

But

$$B_k^{-1}(t) = C_{N,k} t^{\frac{N-2k}{2N}} \quad t \geq 0, \quad C_{N,k} > 0, \quad k \geq 1$$

and from this

$$\int_1^{\infty} \frac{B_N^{-1}(\tau)}{\tau^{\frac{N+1}{N}}} d\tau < \infty.$$

By computing again we find that

$$\int_0^1 \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty, \quad k = 1, 2.$$

Therefore by theorem A.1 we have

$$W^2 L_{B_0}(\Omega) \hookrightarrow L_A(\Omega),$$

since as we have shown above $J \geq 2$ and yet by (1.4)

$$\frac{B_2(\lambda t)}{A(t)} = \frac{C_{N,\lambda}|t|^{2^*}}{A(t)} \rightarrow \infty \quad \text{as } t \rightarrow \infty, \quad \lambda > 0.$$

The case $m = 1$ that is

$$W^1 L_{B_0}(\Omega) \hookrightarrow L_A(\Omega)$$

is similar and even more direct.

Hence

$$W^m L_{B_0}(\Omega) \hookrightarrow L_A(\Omega) \quad m = 1, 2.$$

Using (3.2) we finally get

$$H_m \hookrightarrow L_A(\Omega) \quad m = 1, 2.$$

This completes the proof of lemma 4.

Before proceeding to the proof of lemma 5 we consider the function $a^*(t) \equiv 2C_2 a(t)$. We remark that $a^*(t)$ has the same properties of $a(t)$ and in addition its potential $A^*(t) \equiv \int_0^t a^*(\tau) d\tau$ is an N-function having the same properties as $A(t)$. In particular A^* satisfies (1.4) and moreover

$$|f(x, t)| \leq C_1 + \frac{1}{2} a^*(t).$$

PROOF OF LEMMA 5.

Using (1.7) we have

$$C \geq \frac{1}{2} \|u\|_m^2 - \int_{\Omega} F(x, u_n) \geq \frac{1}{2} \|u\|_m^2 - C - \frac{1}{\theta} \int_{\Omega} u_n f(x, u_n). \quad (3.7)$$

Now since $I'_m(u_n) \rightarrow 0$ we have

$$|\langle I'_m(u_n), u_n \rangle| \leq \epsilon \|u\|_m \quad \text{for largen}$$

that is

$$\|u\|_m^2 - \int_{\Omega} u_n f(x, u_n) \leq \epsilon \|u\|_m \quad \text{for largen.}$$

Hence

$$\begin{aligned} C &\geq \frac{1}{2} \|u_n\|_m^2 - C - \frac{1}{\theta} \|u_n\|_m^2 - \frac{1}{\theta} \epsilon \|u_n\|_m \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_m^2 - \frac{1}{\theta} \epsilon \|u_n\|_m - C \end{aligned}$$

showing that u_n is bounded in H_m . Hence by lemma 4 there is some $u \in H_m$ such that

$$u_n \rightharpoonup u \text{ in } H_m \quad \text{and} \quad u_n \rightarrow u \text{ in } L_{A^*}(\Omega).$$

On the other hand, since $I'_m(u_n) \rightarrow 0$ we have

$$\langle u_n, \phi \rangle_m - \int_{\Omega} f(x, u_n) \phi = o(1), \quad \phi \in H_m.$$

We claim that

$$|f(x, u_n)|_{L_{A^*}} \leq C, \text{ for some } C > 0. \quad (3.8)$$

Assume (3.8) for a while. Using Hölder inequality in Orlicz spaces for L_{A^*} and $L_{\bar{A}^*}$ where $\bar{A}^*$ is the conjugate function of A^* (see e.g. Adams [1, pg 234]) we get

$$|\langle u_n, \phi \rangle_m| \leq o(1) + |f(x, u_n)|_{L_{A^*}} |\phi|_{L_{\bar{A}^*}} \quad (3.9)$$

Now replacing ϕ by $u_n - u$ in (3.9) and using (3.8) we have

$$0 = \lim \langle u_n, u_n - u \rangle_m = \lim \langle u_n, u_n \rangle_m = \lim \langle u_n, u_n \rangle_m - \langle u, u \rangle_m$$

showing that $u_n \rightarrow u$ in H_m .

VERIFICATION OF (3.8). We have

$$\begin{aligned} \int_{\Omega} \bar{A}^*(|f(x, u_n)|) &\leq \int_{\Omega} \bar{A}^*\left(C_1 + \frac{1}{2}a^*(|u_n|)\right) \\ &\leq \frac{1}{2} \int_{\Omega} \bar{A}^*(2C_1) + \frac{1}{2} \int_{\Omega} \bar{A}^*(a^*(|u_n|)) \\ &\leq C + \frac{1}{2} \int_{\Omega} A^*(|u_n|) + \int_{\Omega} |u_n| a^*(|u_n|) \\ &\leq C + C_1 \left[\int_{\Omega} |u_n|^{p^*} + \int_{\Omega} |u_n|^{p^*} \right] \leq C \end{aligned}$$

showing (3.8) and consequently lemma 5.

PROOF OF THEOREM 1.

We have already shown using the lemmata above that I_m has a critical point $u \in H_m$ so that

$$\langle u, v \rangle_m = \int_{\Omega} f(x, u)v, \quad v \in H_m.$$

In the case $m = 1$, we have $H_1 = H_o^1$ and so u is a weak solution of $(*)_1$.

In the case $m = 2$ it remains to show that $\Delta u = 0$ on $\partial\Omega$. We use here an argument of [4].

By (1.3) and (1.5), we have

$$f(x, u) \in L^{p^*}(\Omega) \text{ with } \frac{1}{p^*} + \frac{1}{p^*} = 1.$$

Letting $g(x) = f(x, u)$ using the fact that $p^* > 2$ it follows that $W \equiv W^{2, p^*}(\Omega) \cap W_0^{1, p^*}(\Omega) \subset H_2$ and we have

$$\int_{\Omega} \Delta u \Delta z = \int_{\Omega} g(x)z, \quad z \in W.$$

Since $g(x) \in L^{p^*}(\Omega)$ there is a unique $w \in W^{2, p^*}(\Omega) \cap W_0^{1, p^*}(\Omega)$ such that

$$\Delta w = g(x), \quad x \in \Omega.$$

Hence

$$\int_{\Omega} \Delta u \Delta z = \int_{\Omega} \Delta w z = \int_{\Omega} w \Delta z, \quad z \in W.$$

On the other hand given $h \in L^{p^*}(\Omega)$, there is a unique $z \in W$, such that

$$\Delta z = h(x), \quad x \in \Omega.$$

Thus

$$\int_{\Omega} (\Delta u - w)h = 0, \quad h \in L^{p^*}(\Omega)$$

showing that

$$\Delta u = w \quad \text{in } \Omega$$

and so

$$\Delta u = 0, \quad \text{on } \partial\Omega.$$

This proves theorem 1.

4. APPENDIX

At first we recall a general result due to Donaldson & Trudinger [9] (see also Adams [1, theorem 8.40]).

Let C be an N -function and consider the sequence of N -functions

$$B_0(t) \equiv C(t), \quad t \geq 0$$

$$(B_k)^{-1}(t) \equiv \int_0^t \frac{(B_{k-1})^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau, \quad k = 1, 2, \dots, \quad t \geq 0.$$

It follows that

$$\int_1^\infty \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty \quad \text{for some } k \geq 1.$$

Let us label $J \equiv J(C)$ the least such k .

THEOREM A.1. Assume $\Omega \subset \mathbb{R}^N$ is a bounded domain with the cone property. Assume also that

$$\int_0^1 \frac{(B_k)^{-1}(\tau)}{\tau^{\frac{(N+1)}{N}}} d\tau < \infty, \quad k = 1, 2, \dots$$

Then

$$W^m L_{B_0}(\Omega) \rightarrow L_{B_m}(\Omega) \tag{3.10}$$

provided $J \geq m$,

$$W^m L_{B_0}(\Omega) \hookrightarrow L_A(\Omega) \tag{3.11}$$

provided both $J \geq m$ and A is an N -function such that

$$\frac{B_m(\lambda t)}{A(t)} \rightarrow \infty \quad \text{as } t \rightarrow \infty, \quad \lambda > 0.$$

Next we present an example to illustrate our assumptions (1.2) – (1.5).

EXAMPLE A.2. Let $a : [0, \infty) \rightarrow \mathbb{R}$ be given by $a(t) = t^{p^*-1}$ if $0 \leq t < 1$, $a(t) = t^{(p^*-1) - \frac{1}{\log(\log(2))}}$ if $1 \leq t < 3$ and $a(t) = t^{(p^*-1) - \frac{1}{\log(\log(n))}}$ if $n \leq t < (n+1)$ for $n = 3, 4, \dots$

Then a satisfies (1.2), (1.5) and it is a straightforward calculation to show that A satisfies (1.4).

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