DIFFERENCE SEQUENCE SPACES

A.K. GAUR
Department of Mathematics
Duquesne University
Pittsburgh, PA 15282, U.S.A.
and
MURSALEEN*
Department of Mathematics
Aligarh Muslim University
Aligarh 202002, INDIA

(Received April 17, 1996 and in revised form July 29, 1996)

ABSTRACT. In [1]

$$S_r(\Delta) = \{x = (x_k) : (k^r|\Delta x_k|)_k^{\infty} \in c_0\}$$

for $r \geq 1$ is studied. In this paper, we generalize this space to $S_r(p, \Delta)$ for a sequence of strictly positive reals. We give a characterization of the matrix classes $(S_r(p, \Delta), \ell_\infty)$ and $(S_r(p, \Delta), \ell_1)$.

KEY WORDS AND PHRASES: Difference sequence spaces, Köthe-Toeplitz duals, matrix transformations.

1991 AMS SUBJECT CLASSIFICATION CODES: 40H05, 46A45.

1. INTRODUCTION

Let $\ell_\infty$, $c$ and $c_0$ be the sets of all bounded, convergent and null sequences of $x = (x_k)_1^{\infty}$, respectively. Let $w$ denote the set of all complex sequences and $\ell_1$ denote the set of all convergent and absolutely convergent series.

Let $z$ be any sequence and $Y$ be any subset of $w$. Then

$$z^{-1} \cdot Y = \{x \in w : zx = (x_kz)_1^{\infty} \in Y\}.$$

For any subset $X$ of $w$, the sets

$$X^\alpha = \bigcap_{x \in X} (z^{-1} \cdot \ell_1)$$

and

$$X^\beta = \bigcap_{x \in X} (z^{-1} \cdot c\ell)$$

are called the $\alpha$- and $\beta$-duals of $X$.

We define the linear operators $\Delta, \Delta^{-1} : w \to w$ by

$$\Delta x = (\Delta x_k)_1^{\infty} = (x_k - x_{k+1})_1^{\infty},$$

and

$$\Delta^{-1} x = (\Delta^{-1} x_k)_1^{\infty} = \left(\sum_{j=1}^{k-1} x_j\right)_1^{\infty},$$

such that
Let
\[ S_r(\Delta) := \{ x \in w : (k^r|\Delta x_k|)_k=1^\infty \in c_0 \}, \text{ see [1].} \]

In this paper we extend the space \( S_r(\Delta) \) to \( S_r(p, \Delta) \) in the same manner as \( c_0, c, \ell_\infty \) were extended to \( c_0(p), c(p), \ell_\infty(p) \), respectively (cf. [2],[3],[4]). We also determine the \( \alpha \)- and \( \beta \)-duals of our new sequence space. Let \( p = (p_k)_k=1^\infty \) be an arbitrary sequence of positive reals and \( r \geq 1 \), then we define
\[ S_r(p, \Delta) := \{ x \in w : (k^r|\Delta x_k|)_k=1^\infty \in c_0(p) \}, \]
where
\[ c_0(p) := \left\{ x \in w : \lim_{k \to \infty} |x_k|^p_k = 0 \right\}. \]

If \( p = e = (1, 1, 1, \ldots) \), then the set \( S_r(p, \Delta) \) reduces to the set \( S_r(\Delta) \). For \( r = 0 \), \( S_r(p, \Delta) \) is the same as \( \Delta c_0(p) \) (cf. [5],[6],[7]).

We will need the following lemmas:

**Lemma 1** (Corollary 1 in [7]). Let \( (p_k)_{k=1}^\infty \) be a sequence of nondecreasing positive reals. Then \( a \in (p_k)_{k=1}^\infty \cdot c_0 \) implies \( R(n \in (p_n)_{k=1}^\infty \cdot c_0 \) where \( R_n = \sum_{k=n+1}^\infty a_k \) \( n = 1, 2, \ldots \).

**Lemma 2** (Lemma 1(b) in [8]). Let \( p = (p_k)_{k=1}^\infty \) be a strictly positive sequence such that \( p \in \ell_\infty \). Then \( A \in (c_0(p), \ell_1) \) if and only if
\[ (*) \quad B(M) = \sup_{N \text{ finite}} \left( \sum_{k=1}^N \sum_{n \in N} \left| a_{nk} \right| \frac{M^{-1/p_k}}{M} \right) < \infty \]
for some integer \( M \geq 2 \).

2. **THE \( \alpha \)- AND \( \beta \)-DUALS OF \( S_r(p, \Delta) \)**

**Theorem 2.1.** Let \( p = (p_k)_{k=1}^\infty \) be a strictly positive sequence and \( r \geq 1 \). Then
\[ (a) \quad [S_r(p, \Delta)]^\alpha = \bigcup_{N>1} D_r^{(1)}(p), \]
\[ (b) \quad [S_r(p, \Delta)]^\beta = C_r(p) = \bigcap_{\nu \in c_0^+} \bigcap_{N>1} D_r^{(2)}(p) \bigcap_{N>1} D_r^{(3)}(p), \]
where
\[ D_r^{(1)}(p) := (\Delta_r^{-1}N^{-1/p})^{-1} \cdot \ell_1 = \left\{ a \in w : \sum_{k=1}^\infty \sum_{j=1}^{k-1} \frac{a_{jk}}{|R_k|} < \infty \right\}, \]
\[ D_r^{(2)}(p) := (\Delta_r^{-1}v_j^{1/p})^{-1} \cdot c_0 = \left\{ a \in w : \sum_{k=1}^\infty \sum_{j=1}^{k-1} \frac{v_j^{1/p}}{|R_k|} \text{ converges} \right\}, \]
\[ D_r^{(3)}(p) := \left\{ a \in w : R \in \left( \frac{N^{-1/p}}{k^r} \right)^{-1} \cdot \ell_1 \right\} = \left\{ a \in w : \sum_{k=1}^\infty |R_k| \frac{N^{-1/p}}{k^r} < \infty \right\}, \]
and \( c_0^+ \) is the set of all positive sequences in \( c_0 \).

**Proof.** (a) Let \( a \in \bigcup_{N>1} D_r^{(1)}(p) \). Then
\[ a \cdot s(1/N_0) \in \ell_1 \text{ for some } N_0 \geq 2, \quad (2.1) \]
where
\[ s(1/N_0) = \left( s_k \left( \frac{1}{N_0} \right) \right)_{k=1}^{\infty} = \left( \sum_{j=1}^{k} \frac{N_0^{-1/p_j}}{j'} \right)_{k=1}^{\infty}. \]

Since \( s(1/N_0) \) is increasing, (2.1) implies that
\[ a \in \ell_1. \] (2.2)

Let \( x \in S_r(p, \Delta) \). Then for a given \( N_0 \in \mathbb{N} \), there exists an \( M = M(N_0) \in \mathbb{N} \) such that
\[ \sup_{k \geq M} |\Delta x_k|^{p_k} < \frac{1}{N_0}, \] and hence \( |\Delta x_k| \leq \frac{N_0^{-1/p_k}}{k^r} \) for all \( k = 1, 2, \ldots \), and consequently by (2.1) we have
\[ \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} |\Delta x_j| \leq \sum_{k=1}^{\infty} |a_k| s_k(1/N_0) < \infty. \] (2.3)

Finally, by (2.2) and (2.3), we get
\[ a \in [S_r(p, \Delta)]^a. \]

Let \( a \notin \bigcup_{N_1 \geq 1} D_r^{(1)}(p) \). Then we can determine a strictly increasing sequence \( (k(m))_{m=1}^{\infty} \) of integers such that \( k(1) = 1 \) and
\[ \sum_{k=k(m)}^{k(m+1)-1} |a_k| s_k(1/(m + 1)) > 1 \ (m = 1, 2, \ldots). \]

We define the sequence \( x = (x_k) \) by
\[ x_k = \sum_{i=1}^{m} \sum_{j=1}^{\min\{k-1, k(i+1)-1\}} \frac{(i+1)^{-1/p_j}}{j'}, \quad (k(m) \leq k \leq k(m+1)-1; m = 1, 2, \ldots). \]

Then \( x \in S_r(p, \Delta) \) and
\[ \sum_{k=1}^{\infty} |a_k| |x_k| = \sum_{m=1}^{\infty} \sum_{k=k(m)}^{k(m+1)-1} |a_k x_k| > \infty \]
which proves that
\[ a \notin [S_r(p, \Delta)]^a. \]

Hence, \([S_r(p, \Delta)]^a = \bigcup_{n > 1} D_r^{(n)}(p)\).

(b) Let \( a \in C_r(p) \). Then \( a \in cs \), and Abel's summation by parts yields
\[ \sum_{k=1}^{n} a_k x_k = -\sum_{k=1}^{n-1} R_k \Delta x_k + R_n \sum_{k=1}^{n-1} \Delta x_k + x_1 \sum_{k=1}^{n} a_k \text{ for all } x, \ (n = 1, 2, \ldots). \] (2.4)

Further
\[ R \in \left( \frac{N_0^{-1/p}}{k^r} \right) \cdot \ell_1 \text{ for some integer } N_0 \geq 2. \] (2.5)

Let \( x \in S_r(p, \Delta) \). Then there is a sequence \( v \in c_0^+ \) such that
\[ |\Delta x_k| \leq \frac{v_k^{1/p_k}}{k^r} \ (k = 1, 2, \ldots) \text{ and } |\Delta x_k| \leq \frac{N_0^{-1/p_k}}{k^r} \]
for all sufficiently large \( k \). Now, by (2.5)
\[ \sum_{k=1}^{\infty} |R_k| |\Delta z_k| < \infty. \]

Hence

\[ R \Delta x \in \ell_1 \subset cs. \quad (2.6) \]

Finally, by Lemma 1, \( a \in (\Delta^{-1} v^{1/p})^{-1} \cdot cs \) implies that

\[ R \in (\Delta^{-1} v^{1/p})^{-1} \cdot c_0 \quad (2.7) \]

and consequently

\[ R_n \sum_{k=1}^{n-1} \Delta z_k \to 0 \quad (n \to \infty). \quad (2.8) \]

From \( a \in cs, (2.4), (2.6) \) and (2.8), we conclude that

\[ \sum_{k=1}^{\infty} a_k z_k = - \sum_{k=1}^{\infty} R_k \Delta z_k + z_1 \sum_{k=1}^{\infty} a_k \quad (2.9) \]

and \( ax \in cs. \) Thus \( a \in [S_r(p, \Delta)]^d. \) Now, let \( a \in [S_r(p, \Delta)]^d. \) Then \( ax \in cs \) for all \( x \in S_r(p, \Delta) \) and \( v \in S_r(p, \Delta). \) This implies that \( a \in cs. \) Let \( v \in c_0^+ \) be given. Then \( z = \Delta^{-1} v^{1/p} \in S_r(p, \Delta). \) Hence \( a \in (\Delta^{-1} v^{1/p})^{-1} \cdot cs, \) and by Lemma 1, we get (2.7). Therefore (2.8) holds for all \( x \in S_r(p, \Delta). \) By (2.4), we get \( R \Delta x \in cs. \) Since \( x \in S_r(p, \Delta) \) if and only if \( y = \Delta x = (k^n \Delta z_k)_{k=1}^{\infty} \in c_0(p), \) this implies that

\[ \sum_{k=1}^{\infty} \left| R_k \left| \frac{N^{-1/p}}{k^r} \right| \right| < \infty \]

for some integer \( N \geq 2 \) (cf. [9], Theorem 6). Hence \([S_r(p, \Delta)]^d = C_r(p).\)

3. MATRIX TRANSFORMATIONS

For any infinite complex matrix \( A = (a_{nk})_{n,k=1}^{\infty}, \) we write \( A_n = (a_{nk})_{n,k=1}^{\infty} \) for the sequence in the \( n \)th row of \( A. \) Let \( X \) and \( Y \) be two subsets of \( \omega. \) By \((X, Y),\) we denote the class of all matrices \( A \) such that the series \( A_n(x) = \sum_{k=1}^{\infty} a_{nk} z_k \) converges for all \( x \in X \) and each \( n \in N, \) and the sequence \( A x = (A_n(x))_{n=1}^{\infty} \in Y \) for all \( x \in X. \)

**THEOREM 3.1.** Let \( p = (p_k)^{\infty} \) be a strictly positive sequence and \( r \geq 1. \) Then \( A \in (S_r(p, \Delta), \ell_\infty) \) if and only if

(i) \( D_r(v) = \sup_n |A_n(\Delta^{-1} v^{1/p})| = \sup_n \left| \sum_{k=1}^{\infty} a_{nk} \sum_{j=1}^{k-1} \frac{v^{1/p}}{j^r} \right| < \infty \) for all \( v \in c_0^+, \)

(ii) \( D_r(M) = \sup_n \left| R_{nk}\frac{M^{1/p}}{k^r} \right| < \infty \) for some integer \( M \geq 2, \)

where \( R_{nk} = \sum_{j=k+1}^{\infty} a_{nk} \) for all \( n \) and \( k, \) and

(iii) \( D_\infty = \sup_n |A_n(e)| = \sup_n \left| \sum_{k=1}^{\infty} a_{nk} \right| < \infty. \)
PROOF. Let the conditions (i), (ii) and (iii) be true and \( x \in S_r(p, \Delta) \). By Theorem 2.1(b), conditions (i) and (ii) imply that \( A_n \in [S_r(p, \Delta)]^p \) for \( n = 1, 2, \ldots \). For a given \( M \in \mathbb{N} \), there exists a \( M' = M'(M) \in \mathbb{N} \) such that \( \sup_{k \geq M'} |k \tau \Delta x_k| \leq \frac{1}{M} \), where \( M \geq 2 \) is the integer in (ii). By (2.9), we have
\[
|A_n(x)| \leq D_r(M) + |x_1|D_\infty \quad (n = 1, 2, \ldots)
\]
and hence \( Ax \in \ell_\infty \). Conversely, let \( A \in (S_r(p, \Delta), \ell_\infty) \). Since \( x = \Delta_\tau^{-1}v^{1/p} \in S_r(p, \Delta) \) for all \( v \in c_0^r \), condition (i) follows immediately. Also the necessity of (iii) follows from the fact that \( x = e \in S_r(p, \Delta) \). Now, by (i), (iii) and (2.9),
\[
A_n(x) = - \sum_{k=1}^{\infty} R_n k \Delta x_k + x_1 A_n(e) \quad (n = 1, 2, \ldots).
\]
Since \( Ax \in \ell_\infty \) and \( x_1 A_e \in \ell_\infty \), therefore \( (R_n k \Delta x_k)_{n=1}^{\infty} \in \ell_\infty \). Since \( x \in S_r(p, \Delta) \) if and only if \( (k' \Delta x_k)_{k=1}^{\infty} \in c_0(p) \), and \( \left( \sum_{k=1}^{\infty} (R_n k') (k' \Delta x_k) \right)_{n=1}^{\infty} \in \ell_\infty \) for all \( (k' \Delta x_k)_{k=1}^{\infty} \in c_0(p) \), this implies that \( D_r(M) < \infty \) for some integer \( M \geq 2 \), and (ii) holds.

THEOREM 3.2. Let \( p = (p_k)_{k=1}^{\infty} \) be a strictly positive sequence such that \( p \in \ell_\infty \), and \( r \geq 1 \). Then \( A \in (S_r(p, \Delta), \ell_1) \) if and only if

(i) \[ C_r^{(1)}(v) := \sup_{N \in \mathbb{N} \text{ finite}} \sum_{n \in N} A_n(\Delta_\tau^{-1}v^{1/p}) < \infty \]

for all sequences \( v \in c_0^r \),

(ii) \[ C_r^{(2)}(M) := \sup_{N \in \mathbb{N} \text{ finite}} \left( \sum_{k=1}^{\infty} \sum_{n \in N} a_{nk} k^{-1} \sum_{j=1}^{v^{1/p}} \frac{v^{1/p}}{j} \right) < \infty \]

for some integer \( M \geq 2 \), and

(iii) \[ D_r^{(3)} := \sup_{N \in \mathbb{N} \text{ finite}} \sum_{n \in N} A_n(e) \quad < \infty. \]

PROOF. Let conditions (i), (ii) and (iii) hold. Then \( A_n \in [S_r(p, \Delta)]^p \). Let \( x \in S_r(p, \Delta) \). For a given \( M \in \mathbb{N} \) there exists a \( M' = M'(M) \in \mathbb{N} \) such that \( \sup_{k \geq M'} |k \tau \Delta x_k|^{1/p} < \frac{1}{M} \). Now, by (2.9) and the inequality in [10], p. 33, we have
\[
\sum_{n=1}^{m} |A_n(x)| \leq 4(C_r^{(2)}(M) + |x_1|D_r^{(3)}) < \infty.
\]
Since \( m \in N \) is arbitrary, we have \( Ax \in \ell_1 \). Conversely, let \( A \in (S_r(p, \Delta), \ell_1) \). Then
\[
\left| \sum_{n \in N} A_n(x) \right| \leq \sum_{k=1}^{\infty} |A_n(x)| < \infty
\]
for all \( x \in S_r(p, \Delta) \) and for all finite subsets \( N \) of \( N \). Therefore the necessity of (iii) and (i) follows immediately, since \( e \) and \( x = \Delta_\tau^{-1}v^{1/p} \in S_r(p, \Delta) \) for every sequence \( v \in c_0^r \). Further we have
\[
\left( \sum_{k=1}^{\infty} \frac{R_{nk}}{k^r} k^n \Delta x_k \right)_{n=1}^{\infty} \in \ell_1 \quad \text{for all} \quad (k^n \Delta x_k)_{k=1}^{\infty} \in c_0(p),
\]

and hence (ii) holds by Lemma 2.

ACKNOWLEDGMENT. ({\ast}) This research is supported by the University Grant Commission, number F.8-14/94. The authors are grateful to the referee for his or her valuable suggestions which improved the clarity of this presentation.

REFERENCES


Submit your manuscripts at http://www.hindawi.com