RESEARCH NOTES

A SUBSET OF METRIC PRESERVING FUNCTIONS

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ABSTRACT. In this paper we define a subset of metric preserving functions and give some examples and a characterization of this subset.

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1. INTRODUCTION

We call a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ a metric preserving function if and only if $f(\rho) : M \times M \to \mathbb{R}^+$ is a metric for every metric $\rho : M \times M \to \mathbb{R}^+$, where $(M, \rho)$ is an arbitrary metric space and $\mathbb{R}^+$ denotes the nonnegative reals. We will denote the collection of metric preserving functions by $\mathcal{M}$. There are many papers out there which deal with these functions (see the references). Of particular interest is the derivative of metric preserving functions. In [1] J. Boráš and J. Doboš show that if $f \in \mathcal{M}$ is differentiable then $|f'(x)| \leq f'(0)$. J. Doboš and Z. Piotrowski in [2] construct two examples concerning differentiation and metric preserving functions. The first $f \in \mathcal{M}$ is continuous and nowhere differentiable. The other is metric preserving, differentiable and the derivative is infinite exactly on $\{0\} \cup 2^{-n}, n = 1, 2, 3, \ldots$. In [9] this author answers a question of Doboš and Piotrowski by showing how for any measure zero, $\lambda$ set in $[0, \infty)$ there is a continuous metric preserving function whose derivative is infinite on that set union zero.

The subset of metric preserving functions we wish to consider is defined below.

DEFINITION. Let $f \in \mathcal{M}$ be differentiable on $(0, \infty)$. Define $g(x)$ as

$$g(x) = \begin{cases} f'(x) & x \in (0, \infty) \\ 0 & x = 0 \end{cases}$$

We say $f \in \mathcal{D}$ if and only if $f, g \in \mathcal{M}$.

The purpose of this paper is to give examples of these types of functions and to characterize the type of $f$ which can be in $\mathcal{D}$.

2. MAIN RESULTS

We note here that the set $\mathcal{D}$ is nonempty. It is easy to see that $\mathcal{D}$ contains all functions of the form $f(x) = kx, k > 0$. A natural question to then ask is if it is possible that there are functions $f$ such that $g$ defined above is continuous at the origin (which is not the case for $f(x) = kx$). The answer is no and is given in the following theorem.

THEOREM 1. If $f$ is differentiable on $[0, \infty)$ and metric preserving $f'(x)$ is not a metric preserving function.

PROOF. If $f' \in \mathcal{M}$ then $f'(0)$ would have to be zero and $f' > 0$ on $(0, \infty)$ implies there must be some $[0, \varepsilon)$ where $f$ must be strictly convex. Then $f \notin \mathcal{M}$ from Prop. 10 in [1].

Nor can we go in the opposite direction and assume that if $g$ is metric preserving its integral will also be metric preserving.

EXAMPLE. There exists a metric preserving function $g$ whose integral, $\int_0^x g(t) dt$, is not also metric preserving.

PROOF. Let $g(x) = 1 - e^{-x}$. Then $\int_0^x 1 - e^{-t} dt$ is strictly convex in a neighborhood of the origin.
Note that \( g(x) = 2x \) would also serve in the example above. While both are continuous, \( 1 - e^{-x} \) has the added strength of being bounded. We now can look at some properties of these functions in \( D \).

**THEOREM 2.** If \( f \in D \), \( f \) is nondecreasing.

**PROOF.** This is a consequence of the fact that the function \( g(x) \) must be greater than zero since \( g \) is metric preserving.

**LEMMA.** Let \( f \in \mathcal{M} \) and \( \limsup_{x \to 0^+} f(x) = a \). Then for all \( x \in [0, \infty) \), \( f(x) \geq a/2 \).

**PROOF.** This is a property of \( f \) being metric preserving. See Corollary 1 in [1].

**THEOREM 3.** Let \( f(x) = x^k \). Only \( f \in D \) if and only if \( k = 1 \).

**PROOF.**

If \( k > 1 \) then \( f \notin \mathcal{M} \) since \( f \) would be strictly convex around the origin.

If \( k \in (0, 1) \) then \( g \) violates the lemma above.

If \( k = 0 \) then \( g \notin \mathcal{M} \) since \( g \) would be identically zero.

If \( k < 0 \) then \( f \) violates the lemma above.

In order to characterize functions in the set \( D \) we need the notion of a triangle triplet. The 3-tuple \((a, b, c) \in (\mathbb{R}^+)^3\) is called a triangle triplet if \( a \leq b + c \), \( b \leq a + c \), and \( c \leq a + b \). This is another way to determine if \( f \) is a metric preserving function (see F. Terpe [8]). A function \( f \) is a metric preserving function if and only if \( f(0) = 0 \) and \((f(a), f(b), f(c))\) is a triangle triplet whenever \((a, b, c)\) is one. This gives us a way to describe these functions in \( D \).

**THEOREM 4.** Let \( g(x) : \mathbb{R}^+ \to \mathbb{R}^+ \) be a function satisfying

\[
\forall a > 0 \int_0^a g(x)dx \geq \int_a^c g(x)dx \quad \text{where} \quad c - b = a. \tag{2.1}
\]

If there exists an \( A > 0 \) such that

\[
A \leq N + Mg(x) \leq 2A, \tag{2.2}
\]

then both \( G(x) = \begin{cases} N + Mg(x) & x > 0 \\ 0 & x = 0 \end{cases} \) and \( F(x) = \int_0^x G(t)dt \) are in \( \mathcal{M} \).

**PROOF.** The condition (2.2) gives us \( G(x) \) is metric preserving (Proposition 3 in [1]). Condition (2.1) assures that \( F(x) \) will satisfy the triangle triplet condition. Assume \( a < b < c \). Then \( F(a) \leq F(b) + F(c) \) and \( F(b) \leq F(a) + F(c) \) are automatic. Lastly,

\[
F(c) = F(b) + \int_b^c G(t)dt \leq F(b) + \int_0^b G(t)dt = F(a) + F(b). \tag{2.3}
\]

This describes such examples in \( D \) using \( 1 + e^{-x} \), \( 3 + \frac{1}{2} \cos(1/x) \), and \( 3 + e^{-x} \cos x \) for \( N + Mg(x) \).

To close we note that this gives another way to create metric preserving functions.

**COROLLARY.** If \( g(x) \) meets condition (2.1) and \( 0 \leq g(x) \) almost everywhere then \( g(x) \) need not be in \( \mathcal{M} \), but \( \int_0^\infty g(t)dt \lambda \) is in \( \mathcal{M} \) where \( \lambda \) denotes Lebesgue measure.

**REFERENCES**


