FUNCTIONAL EVOLUTION EQUATIONS WITH NONCONVEX LOWER SEMICONtinuous MULTIVALUED PERTURBATIONS

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ABSTRACT. In this paper we prove some existence theorems concerning the solutions and integral solution for functional (delay) evolution equations with nonconvex lower semicontinuous multivalued perturbations.

KEY WORDS AND PHRASES: Functional evolution equations, m-accretive operators, integral solutions

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1. INTRODUCTION

Let $E$ be a Banach space, $r, T \in \mathbb{R}^+$ and $I = [a, b]$. Let us denote $C_E([-r, T])$ the vector space of all continuous functions from $[-r, T]$ to $E$ endowed with the uniform topology.

For all $t \geq 0$, $s_t : C_E([-r, t]) \to C_E([-r, 0])$,

$$(s_t f)(\theta) = f(t + \theta), \quad \forall \theta \in [-r, 0].$$

$A : I \times E \to 2^E$ such that $A(t,.)$ is an m-accretive multivalued operator.

$P_{wc}(E)$ the family of nonempty weakly compact subsets of $E$.

In this paper we are concerned with the following problems:

1. Existence of solutions of the perturbated evolution equation with delay

$$\begin{cases}
    u'(t) \in \quad A(t, u(t)) + F(t, s_t u) & \text{a.e. on } I, \\
    u \equiv \psi & \text{on } [-r, 0]
\end{cases}$$

where $F : I \times C_E([-r, 0]) \to P_{wc}(E)$ is a multivalued function such that $F(t,.)$ is lower semicontinuous and $\psi \in C_E([-r, 0])$ is arbitrary but fixed.

2. Existence of solutions of the perturbated evolution equation with delay

$$\begin{cases}
    u'(t) \in \quad N_{\Gamma(t)}(u(t)) + F(t, s_t u) & \text{a.e. on } I, \\
    u \equiv \psi & \text{on } [-r, 0]
\end{cases}$$

where $N_{\Gamma(t)}(x)$ is the normal cone of the convex set $\Gamma(t)$ at the point $x \in E; t \in I$. It should be noticed that the problem (Q) is not a special case of the problem (P).

3. Existence of integral solutions of (P), when the operator $A$ is independent of $t$, under conditions that are weaker than those imposed in (P).

The results obtained in the present paper generalized the following interesting known cases:

Problem (P) for which the dual of $E$ is uniformly convex, $A(t,.)$ is an m-accretive single-valued operator and $F$ is a Lipschitz single-valued function \text{ cf.} Kartsatos and Parrott [1].
Problem \((P)\) for which \(E\) is reflexive, \(A(t, \cdot)\) is an \(m\)-accretive multivalued operator and \(F\) is a Lipschitz single-valued function cf Tanaka \[2\].

Problems \((P)\) and \((Q)\) without delay cf Cichon \[3\], \[4\], Ibrahim \[5\] and the references therein.

2. NOTATIONS AND DEFINITIONS

Let \(E^*\) be the dual of \(E\), \(E_\circ\) the Banach space \(E\) endowed with the weak topology \(\sigma(E, E^*)\) If \(B\) is a multivalued operator from \(E\) to \(2^E\) then \(B\) is said to be accretive if for each \(\lambda > 0\), \(x_1, x_2 \in D(B)\) (the domain of \(B\)), \(y_1 \in B(x_1)\) and \(y_2 \in B(x_2)\) we have
\[
\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|.
\]
We say that \(B\) is \(m\)-accretive if \(B\) is accretive and if there exists \(\lambda > 0\) such that \(R(I + \lambda B) = E\), where \(I\) is the identity map. It is known that if \(B\) is \(m\)-accretive, then for every \(\lambda > 0\) the resolvent \(J_\lambda B = (I + \lambda B)^{-1}\) and the Yosida approximation of \(B, B_\lambda = (I - J_\lambda B)/\lambda\), are defined everywhere.

The generalized domain of \(B\) is defined by
\[
D'(B) = \{x \in E : \|B(x)\| = \lim_{\lambda \to \infty} \|B_\lambda x\| < \infty\}.
\]
For the properties of \(m\)-accretive multivalued operators refer to \[6\] and \[7\].

If \(C\) is a convex subset of \(E\) and \(x \in C\), then the normal cone of \(C\) at \(x\) is defined by
\[
N_C(x) = \{y \in E^* : \langle y, z - x \rangle \leq 0, \forall z \in C\}.
\]

Now we recall some concepts concerning multivalued functions. Let \(Y\) be a locally convex space and let \(G : E \to 2^Y - \{\phi\}\). We say that \(G\) is lower semicontinuous (resp. upper semicontinuous) if for every open \(V\) in \(Y\) the set \(\{x \in E : G(x) \cap V \neq \phi\}\) (resp \(\{x \in E : G(x) \subset V\}\) is open in \(E\). We say that \(G\) is lower semicontinuous (resp. upper semicontinuous) in the Kuratowski sense iff for all \(v_n \to v\) in \(E\), \(G(v) \subseteq \lim_{n \to \infty} \inf G(v_n)\) (resp \(\lim_{n \to \infty} \sup G(v_n) \subseteq G(v)\)), where
\[
\begin{align*}
\lim_{n \to \infty} \inf G(v_n) &= \{z \in Y : z = \lim_{n \to \infty} z_n, z_n \in G(v_n), \forall n \geq 1\}, \\
\lim_{n \to \infty} \sup G(v_n) &= \{z \in Y : z = \lim_{n \to \infty} z_n, z_n \in G(v_n), \forall k \geq 1\}.
\end{align*}
\]
If \(E\) is metrizable then lower semicontinuity and lower semicontinuity in the Kuratowski sense are equivalent (cf \[8\], \[9\]).

The following known result will be used in the sequel

**Lemma 3.1** [6]. For every \(t \in I\), let \(A(t, \cdot)\) be an \(m\)-accretive multivalued operator from \(E\) to \(2^E - \{\phi\}\) satisfying the following condition:

\((C_1)\) There exist \(\lambda_0 > 0\), a continuous function \(h : I \to E\) and a nondecreasing continuous function \(L : [0, \infty) \to [0, \infty)\) such that for all \(\lambda \in (0, \lambda_0)\) and for almost \(t, s \in I\),
\[
\|A_\lambda(t, x) - A_\lambda(s, x)\| \leq \|h(s) - h(t)\|L(\|x\|), \quad \forall x \in E.
\]
Then \(D^*(A(t, \cdot))\) and \(\overline{D}(A(t, \cdot))\) are independent of \(t\).

So if \(A\) is as in Lemma 2.1 we may write \(D^*(A(t, \cdot)) = D^*(A(t, \cdot))\) and \(\overline{D}(A(t, \cdot)) = \overline{D}(A(t, \cdot))\); \(t \in I\) respectively.

**Lemma 2.2** [10]. Let \(E\) be a Banach space and \(M\) a compact metric space. If \(T\) is a lower semicontinuous multivalued function on \(M\) and with nonempty closed decomposable values in \(L^1_\mu(I)\), then \(T\) has a continuous selection.

3. EXISTENCE OF SOLUTIONS FOR THE PROBLEMS \((P)\) AND \((Q)\)

To prove our results we need the following lemmas

**Lemma 3.1.** Let \(\psi\) be an element of \(C_E([-\tau, 0])\) and \(\beta\) be a positive real number. The set
\[ \chi = \left\{ u \in C_E([-r, 0]) : u \equiv \psi \text{ on } [-r, 0] \text{ and } u(t) = \psi(0) + \int_0^t f(s)ds; f \in K_\beta \right\}, \]

is nonempty and convex, where \( K_\beta = \{ f \in L^2_E(I) : |f(t)| \leq \beta \text{ a.e. on } I \} \) If \( E \) is reflexive then \( \chi \) is compact subset of \( C_E([-r, T]) \) If, in addition, \( E \) is separable then \( \chi \) is metrizable

**PROOF.** It is obvious that \( \chi \) is nonempty, convex and equicontinuous and that the set \( \{ u(t) : u \in \chi \}; t \in I \), is bounded So, if \( E \) is reflexive then, \( \chi \) is relatively compact in \( C_E([-r, T]) \) by Ascoli's theorem Let us verify that \( \chi \) is closed in \( C_E([-r, T]) \) Let \( (u_n) \) be a sequence in \( \chi \) converging to \( u \in C_E([-r, T]) \) Then \( u \equiv \psi \text{ on } [-r, 0] \) and for each \( n \geq 1 \) there exists \( f_n \in K_\beta \) such that \( u_n(t) = \psi(0) + \int_0^t f_n(s)ds; t \in I \) Since \( E \) is reflexive, \( K_\beta \) is weakly compact in \( L_E^2(I) \) Hence, the sequence \( (f_n) \) has a subsequence, denoted again by \( (f_n) \), converging weakly to \( f \in K_\beta \) Then \( u(t) = \psi(0) + \int_0^t f(s)ds; t \in I \) This proves that \( \chi \) is closed in \( C_E([-r, T]) \) Now if \( E \) is separable then so is \( L_E^2(I) \) Consequently, \( K_\beta \) is metrizable Since \( \chi \) is isomorphic to \( \{ \psi(0) \} \times K_\beta \), then \( \chi \) is metrizable

**LEMMA 3.2.** Let \( G \) be a multivalued function from \( E_0 \) to the nonempty closed subsets of \( E \) such that \( G \) is lower semicontinuous in the Kuratowski sense. If \( (x_n) \) is a sequence converging to \( x \) in \( E_0 \), then for every \( z \in E \),

\[ \limsup_{n \to \infty} d(z, G(x_n)) \leq d(z, \liminf_{n \to \infty} \inf G(x_n)) \leq d(z, G(x)). \]

**PROOF.** Let \( y \in \lim_{n \to \infty} \inf G(x_n) \) Then there exists a sequence \( (y_n) \) such that \( y_n \in G(x_n); n \geq 1 \) and \( y_n \to y \) as \( n \to \infty \). For any \( z \in E \) we have

\[ \limsup_{n \to \infty} d(z, G(x_n)) \leq \limsup_{n \to \infty} \| z - y_n \| = \| z - y \|, \]

which proves the first inequality The second inequality follows from the lower semicontinuity of \( G \)

**THEOREM 3.1.** Let \( E \) be a reflexive separable Banach space Let \( A(t, .); t \in I \) be an \( m \)-accretive multivalued operator from \( E \) to \( 2^E - \{ \phi \} \) satisfying condition \( (C_1) \) together with the following conditions

\[ (C_2) \text{ There exist } \mu > 0 \text{ such that for all } x \in E, \text{ the function } w_x : t \to (I + \lambda A(t, .))^{-1} \text{ belongs to } L_E^2(I) \]

\[ (C_3) \text{ For all } r > 0 \text{ there exists } \delta(r) > 0 \text{ such that for all } \lambda > 0 \text{ and all } x \in \overline{D}(A) \text{ with } \| x \| < r, \]

\[ \| J_A 0, x - x \| \leq \lambda \delta(r). \]

Let \( F \) be a measurable multivalued function from \( I \times C_E([-r, 0]) \) to \( P_{ac}(E) \) satisfying the following conditions

\[ (F_1) \text{ There exists } \alpha > 0 \text{ such that } \]

\[ \sup\{ \| y \| : y \in F(t, u) \} \leq \alpha, \quad \forall (t, u) \in I \times C_E([-r, 0]). \]

\[ (F_2) \text{ For all } t \in I, F(t, .) \text{ is lower semicontinuous in the sense of Kuratowski from } C_E([-r, 0]) \text{ to } E \]

\[ (F_3) \text{ For all } u \in C_E([-r, 0]) \text{ the multivalued function } t \to F(t, s_t u) \text{ admits a measurable selection } \]

Then for every \( \psi \in C_E([-r, 0]) \) with \( \psi(0) \in D^*(A) \), the problem \( (P) \) has a solution

**PROOF.** We split the proof into the following three steps

**(1)** Let \( f \in K_\alpha = \{ g \in L^2_E(I) : \| g(t) \| \leq \alpha \text{ a.e. on } I \} \). Since \( A \) satisfies conditions \( (C_1), (C_2) \) and \( (C_3) \), then by Theorem 4 of [5], there exists a unique absolutely continuous function \( u_f : I \to E \) such that

\[ (i) \quad u_f'(t) \in - A(t, u_f(t)) + f(t) \text{ a.e. on } I, \quad u_f(0) = \psi(0), \]

\[ (ii) \quad \| u_f'(t) \| \leq \beta_1 = (\alpha + 1)T + L(\alpha) \sup_{t \in I} \| h(t) \| + \delta(r), \quad \forall t \in I, \text{ where } \]

\[ r = \alpha (1 + L(\| \psi(0) \|)) + \| A(0, x_0) \|, \]
(iii) the function \( f \mapsto u_f \) is continuous from \( K_\alpha \) to \( C_E(I) \)

(2) Set \( \chi_1 = \{ u \in C_E([-r, T]) : u(t) = \psi(t) + \int_0^t f(s) \, ds, f \in K_\alpha \} \). By Lemma 3.1, \( \chi_1 \) is a compact subset of \( C_\alpha([-r, T]) \) and is metrizable. Define a multivalued function \( T_1 \) on \( \chi_1 \) by \( T_1(u) = \{ f \in K_\alpha : f(t) \in F(t, s_1, u) \text{ a.e. on } I \} \). In this step we prove that \( T_1 \) has a continuous selection \( V_1 : \chi_1 \rightarrow K_\alpha \). For this purpose, we show that \( T_1 \) satisfies the conditions of Lemma 2.2. Condition (F3) assures that the values of \( T_1 \) are nonempty. Moreover, if \( D \) is a measurable subset of \( I \) and \( g_1, g_2 \in T_1(u) \) for some \( u \in \chi_1 \), then the function \( g = N_D g_1 + N_{I-D} g_2 \) belongs to \( T_1(u) \), where \( N \) is the characteristic function. Then the values of \( T_1 \) are decomposable. It remains to prove that \( T_1 \) is lower semicontinuous. Since \( \chi_1 \) is compact metrizable in \( C_E([-r, T]) \), it suffices to show that \( T_1 \) is lower semicontinuous in the Kuratowski sense. So, let \( (u_n) \) be a sequence in \( \chi_1 \) converging to \( u \in \chi_1 \), with respect to the topology on \( C_E([-r, T]) \) and let \( g \in T_1(u) \). Since \( F \) is measurable, then for all \( n \geq 1 \) the multivalued function

\[
B_n(t) = \{ z \in F(t, s_1, u_n) : \| g(t) - z \| = d(g(t), F(t, s_1, u_n)) \}
\]

has a measurable selection \( g_n : I \rightarrow E \). Thus, by Lemma 3.2, for all \( t \in I \),

\[
\lim_{n \to \infty} \| g(t) - g(t_n) \| \leq \limsup_{n \to \infty} \sup d(g(t), F(t, s_1, u_n)) \\
\leq d(g(t), \liminf_{n \to \infty} F(t, s_1, u_n)) \\
= d(g(t), F(t, s_1, u)) = 0.
\]

This means that \( T_1 \) is lower semicontinuous and hence there exists a continuous function \( V_1 : \chi_1 \rightarrow K_\alpha \) such that \( V_1(x) \in T_1(x) \) for all \( x \in \chi_1 \).

(3) Define a function \( \theta : \chi_1 \rightarrow \chi_1 \) by \( \theta(x) = u_f = V_1(x) \). By (iii) of the first step, \( \theta \) is continuous. Hence, by Tichonoff's fixed point theorem, there exists \( u \in \chi_1 \) such that \( u = u_f, f = V_1(u) \in T_1(u) \). This means that \( u' \in A(t, u(t)) + f(t) \) and \( f(t) \in F(t, s_1, u) \text{ a.e. on } I \). The theorem is thus proved.

**Theorem 3.2.** Let \( H \) be a Hilbert space and \( F \) be a measurable multivalued function from \( I \times C_H([-r, 0]) \) to \( P(H) \) satisfying conditions (F1), (F2) and (F3). Let \( \Gamma \) be a multivalued function from \( I \) to the family of nonempty closed convex subsets of \( H \), with compact graph \( G \) and satisfies the following conditions.

(\( \Gamma_1 \)) There exists \( \gamma > 0 \) such that \( \| x - \text{proj}_{\Gamma(t)} x \| \leq \gamma (\tau - t) \) for all \( (t, x) \in G \) and all \( \tau \in I, (t < \tau) \)

(\( \Gamma_2 \)) The function \( (t, x) \mapsto \delta^\gamma(t, x, \Gamma(t)) = \sup \{ (x, y) : y \in \Gamma(t) \} \) is lower semicontinuous on \( I \times B_\gamma \), where \( B_\gamma \) is the relative weak topology.

Then for all \( \psi \in C_E([-r, 0]) \) with \( \psi(0) \in \Gamma(0) \), the problem \( (Q) \) has a solution.

**Proof.** We split the proof into the following three steps.

(1) Let \( f \in K_\alpha \). Since \( \Gamma \) has a compact graph and satisfies conditions (\( \Gamma_1 \)) and (\( \Gamma_2 \)) then by Theorem 3.1, there exists a unique absolutely continuous function \( u_f : I \rightarrow H \) such that

\[
\begin{align*}
(i) & \quad u_f(t) = -N_{\Gamma(t)}(u_f(t)) + f(t) \text{ a.e. on } I, \\
(ii) & \quad u_f(0) = \psi(0), u_f(t) \in \Gamma(t), \forall t \in I, \\
(iii) & \quad \| u_f(t) \| \leq \beta_2 = T(\gamma + \alpha), \forall t \in I \text{ and the function } f \mapsto u_f \text{ is continuous from } K_\alpha \text{ to } C_H.
\end{align*}
\]

(2) Set \( \chi_2 = \{ u \in C_H([-r, T]) : u = \psi \text{ on } [-r, 0] \text{ and } u(t) = \psi(t) + \int_0^t f(s) \, ds, f \in K_\alpha \} \) and define a multivalued function \( T_2 \) on \( \chi_2 \) by \( T_2(u) = \{ f \in K_\alpha : f(t) \in F(t, s_1, u) \text{ a.e. on } I \} \). As in the second step of the proof of Theorem 3.1, we can show that \( T_2 \) has a continuous selection \( V_2 : \chi_2 \rightarrow K_\alpha \).

(3) Define the function \( \theta : \chi_2 \rightarrow \chi_2 \) by \( \theta(x) = u_f, f = V_2(x) \). As in the third step of the proof of Theorem 3.1, we can show that there exists a unique \( u \in \chi_2 \) such that \( u = u_f, f \in T_2(u) \). Clearly \( u \) is a solution of \( (Q) \).
4. EXISTENCE OF INTEGRAL SOLUTIONS FOR THE PROBLEM (P) WHEN THE OPERATOR $A$ IS INDEPENDENT OF TIME

In this section $A$ denotes a multivalued operator from $E$ to $2^E - \{\phi\}$. Consider the evolution equation

$$
(P^*) \begin{cases}
    u'(t) \in -A(u(t)) + f(t) & \text{a.e. on } I \\
    u(0) = x_0 \in \overline{D(A)}
\end{cases}
$$

where $f \in L^1_b(I)$. By an integral solution of $(P^*)$ we mean a continuous function $u : I \to \overline{D(A)}$ with $u(0) = x_0$ such that

$$
\|u(t) - z\| \leq \|u(s) - z\| + \int_s^t [u(r) - z, f(r) - y]_+ dr,
$$

for each $z \in D(A), y \in A(z)$ and $0 \leq s \leq t \leq T$, where

$$
[x_1, x_2]_+ = \lim_{h \to 0} (\|x_1 + hx_2\| - \|x_1\|)/h, \forall x_1, x_2 \in E.
$$

It is known that [7] if $A$ is an $m$-accretive operator then for each $(x_0, f) \in \overline{D(A)} \times L^1_b(I)$, the problem $(P^*)$ has a unique integral solution $u_f$, such that the function $f \mapsto u_f$ is continuous. In this section we are concerned with the existence of integral solutions of the functional evolution equation

$$
(P^{**}) \begin{cases}
    u'(t) \in -A(u(t)) + F(t, su(t)) & \text{a.e. on } I \\
    u \equiv \psi & \text{on } [-r, 0]
\end{cases}
$$

where $F$ is a multivalued function from $I \times C_E([-r, 0])$ to $2^E - \{\phi\}$, $S_t : t > 0$ is the operator of translation defined in section 1 and $\psi$ is a given function, belongs to $C_E([-r, 0])$ with $\psi(0) \in \overline{D(A)}$. By an integral solution of $(P^{**})$ we mean a continuous function $u : [-r, T] \to E$ with $u \equiv \psi$ on $[-r, 0]$, such that $u$ is an integral solution of the evolution equation $u'(t) \in -A(u(t)) + f(t), u(0) = \psi(0)$, where $f \in L^1_b(I)$ and $f(t) = F(t, su(t))$, $a.e. on I$.

We say that the operator $A : E \to 2^E - \{\phi\}$ has the $(M)$-property ([7], [12]) if for each $x_0 \in D(A)$ and each uniformly integrable subset $Q$ of $L^1_b(I)$, the set $\{u_g : g \in Q\}$ is a relatively compact subset of $C_E(I)$ where $u_g$ is the unique integral solution of the evolution equation $u'(t) = -A(u(t)) + g(t)$ $a.e.$ on $I$; $u(0) = x_0$. It is well known that ([7], [12]) if the proper operator $-A$ generates a compact semigroup (via Crandall-Liggett’s exponential formula [3], [13]), then $A$ has the property $(M)$.

**THEOREM 4.1.** Let $E$ be a Banach space and $A$ an $m$-accretive multivalued operator from $E$ to $2^E - \{\phi\}$ having the $(M)$-property. Let $F$ be a measurable multivalued function from $I \times C_E([-r, 0])$ to the non-empty closed subsets of $E$ satisfying the condition $(F_3)$ together with the following conditions

- $(F_4)$ There exists a function $h \in L^1_b(I)$ such that

$$
\sup \{\|z\| : z \in F(t, u)\} \leq h(t), \forall (t, u) \in I \times C_E([-r, 0]).
$$

- $(F_5)$ For all $t \in I, F(t, u) : C_E([-r, 0]) \to E$ is lower semicontinuous in the Kuratowski sense

Then for all $\psi \in C_E([-r, 0])$ with $\psi(0) \in \overline{D(A)}$, the problem $(P^{**})$ has an integral solution.

**PROOF.** Consider the set $Q = \{f \in L^1_b(I) : \|f(t)\| \leq h(t) \text{ a.e. on } I\}$. One can easily show that $Q$ is nonempty and uniformly integrable subset of $L^1_b(I)$. As mentioned above, for each $f \in Q$ there exists a unique continuous function $u_f : I \to \overline{D(A)}$ such that $u_f$ is the unique integral solution of the evolution equation $u'(t) = A(u(t)) + f(t), u(0) = \psi(0)$ and the function $f \mapsto u_f$ is continuous from $Q$ to $C_E(I)$. Let $\chi^* = \{u_f^* : u_f^* \in C_E([-r, T]) : f \in Q\}$, where $u_f^* \equiv \psi$ on $[-r, 0]$ and $u_f^* \equiv u_f$ on $I$. Since $a$ has the property $(M), \chi^*$ is compact in the metric space $C_E([-r, T])$. Now, define a multivalued function $T$ on $\chi^*$ by $T(x) = \{f \in L^1_b(I) : f(t) \in F(t, su(t)) \text{ a.e. on } I\}$. As in the second step of the proof of Theorem 3.1, we can show that $T$ has a continuous selection $V : \chi^* \to L^1_b(I)$. 

Also, define a function $\Phi : \mathcal{X} \to \mathcal{X}, \Phi(x) = u^*_f, f = V(x)$. The function $\Phi$ is clearly continuous and hence has a fixed point $x \in \mathcal{X}$. It is obvious that $x$ is the desired solution.

5. EXAMPLES

In this section we give some examples illustrating the scope of the results developed in sections 3 and 4.

**EXAMPLE 1.** Let for all $t \in I$, $A(t) = B - h(t)$ where $h : I \to E$ is integrable and $B$ is an $m$-accretive operator on $E$. Clearly $A(t)$ is $m$-accretive for all $t \in I$. Let $\lambda > 0, s, t \in I$ and $x \in E$. Then

$$\|A_x(t,x) - A_x(s,x)\| \leq \frac{1}{\lambda} \|J_{\lambda}A(t,x) - J_{\lambda}A(s,x)\| \leq \|h(t) - h(s)\|.$$  

Hence condition $(C_1)$ of Lemma 2.1 holds.

**EXAMPLE 2.** In [6] there are several examples for operators $A$ such that for every $t \in I$, $A(t)$ is $m$-accretive and satisfies condition $(C_1)$.

**EXAMPLE 3.** Let $H$ be a real Hilbert space with inner product $(.,.)$ and let $\Phi : H \to H$ be a proper lower semicontinuous convex function. The set $\partial \Phi(x) = \{z \in H : \Phi(x) \leq \Phi(y) + (x - y, z)\}$ for each $y \in H$ is called the subdifferential of $\Phi$ at the point $x$. We recall that $D(\partial \Phi) = \{x \in H : \partial \Phi(x) \text{ is nonempty}\}$. Now if we define an operator $A : D(A) = D(\Phi) \to 2^H$ by $A(x) = \partial \Phi(x)$, then $A$ is $m$-accretive and the following conditions are equivalent [7]:

(i) For each $\lambda > 0$, the resolvent $J_{\lambda}A$ is a compact operator

(ii) The function $\Phi$ is of compact type

(iii) The semigroup generated by the operator $-A$ is compact.

**EXAMPLE 4.** Take $E = L_2([0, \pi])$ and let us define $A : D(A) \subseteq E \to E$ by $Au = -u^{(2)}(t)$ for each $u \in D(A)$ where $D(A) = \{u \in E : u^{(2)} \in E, u(0) = u(\pi) = 0\}$. The operator $A$ is $m$-accretive and the semigroup $\{S(t) : t > 0\}$ generated by $-A(S(t) = \lim_{h \to 0} (I + \frac{t}{h} A)^{-h})$ is compact [7].

REFERENCES


