ASYMPTOTIC THEORY FOR A CRITICAL CASE FOR
A GENERAL FOURTH-ORDER DIFFERENTIAL EQUATION

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ABSTRACT. In this paper we identify a relation between the coefficients that represents a critical case
for general fourth-order equations. We obtained the forms of solutions under this critical case

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1. INTRODUCTION

We consider the general fourth-order differential equation

\[(p_0 y')'' + (p_1 y)' + \frac{1}{2} \sum_{j=0}^{1} \left[ \{q_{2-j} y^{(j+1)}\} + \{q_{2-j} y^{(j+1)}\}^{(j)} \right] + p_2 y = 0\]  \hspace{1cm} (1.1)

where \(x\) is the independent variable and the prime denotes \(d/dx\). The functions \(p_i(x)(0 \leq i \leq 2)\) and \(q_i(x)(i = 1, 2)\) are defined on an interval \([a, \infty)\) and are not necessarily real-valued and are all nowhere
zero in this interval. Our aim is to identify relations between the coefficients that represent a critical case
for (1.1) and to obtain the asymptotic forms of our linearly independent solutions under this case.

Al-Hammadi [1] considered (1.1) with the case where \(p_0\) and \(p_2\) are the dominate coefficients and we
give a complete analysis for this case. Similar fourth-order equations to (1.1) have been considered
previously by Walker [2, 3] and Al-Hammadi [4]. Eastham [5] considered a critical case for (1.1) with
\(p_0 q_2 0\) and showed that this case represents a borderline between situations where all solutions have
a certain exponential character as \(x \to \infty\) and where only two solutions have this character.

The critical case for (1.1) that has been referred, is given by:

\[\frac{q_i'}{q_i} \sim \text{const.} \quad \frac{p_2}{q_2}(i = 1, 2), \quad \left(\frac{p_1 q_1^{-1/2}}{p_1 q_1^{-1/2}}\right) \sim \text{const} \quad \frac{p_2}{q_2}\]  \hspace{1cm} (1.2)

We shall use the recent asymptotic theorem of Eastham [6, section 2] to obtain the solutions of (1.1)
under the above case. The main theorem for (1.1) is given in section 4 with discussion in section 5.

2. A TRANSFORMATION OF THE DIFFERENTIAL EQUATION

We write (1.1) in the standard way [7] as a first order system

\[Y' = AY,\]  \hspace{1cm} (2.1)

where the first component of \(Y\) is \(y\) and
As in [4], we express $A$ in its diagonal form
\[ T^{-1}AT = \Lambda, \] (2.3)
and we therefore require the eigenvalues $\lambda_j$ and eigenvectors $v_j(1 \leq j \leq 4)$ of $A$.

The characteristic equation of $A$ is given by
\[ p_0 \lambda^4 + q_1 \lambda^3 + q_1 \lambda^2 + q_2 \lambda + p_2 = 0. \] (2.4)

An eigenvector $v_j$ of $A$ corresponding to $\lambda_j$ is
\[ v_j = \left( 1, \lambda_j, p_0 \lambda_j^2 + \frac{1}{2} q_1 \lambda_j - \frac{1}{2} q_2 - p_2 \lambda_j^{-1} \right)^t \] (2.5)
where the superscript $t$ denotes the transpose. We assume at this stage that the $\lambda_j$ are distinct, and we define the matrix $T$ in (2.3) by
\[ T = (v_1 \ v_2 \ v_3 \ v_4). \] (2.6)

Now from (2.2) we note that $EA$ coincides with its own transpose, where
\[ E = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \] (2.7)

Hence, by [8, section 2(i)], the $v_j$ have the orthogonality property
\[ (Ev_k)^t v_j = 0 \quad (k \neq j). \] (2.8)

We define the scalars $m_j(1 \leq j \leq 4)$ by
\[ m_j = (Ev_j)^t v_j, \] (2.9)
and the row vectors
\[ r_j = (Ev_j)^t. \] (2.10)

Hence, by [8, section 2]
\[ T^{-1} = \begin{bmatrix} m_1^{-1}r_1 \\ m_2^{-1}r_2 \\ m_3^{-1}r_3 \\ m_4^{-1}r_4 \end{bmatrix}, \] (2.11)
and
\[ m_j = 4p_0 \lambda_j^3 + 3q_1 \lambda_j^2 + 2p_2 \lambda_j + q_2. \] (2.12)

Now we define the matrix $U$ by
\[ U = (v_1 \ v_2 \ v_3 \ v_4) = TK, \] (2.13)
where
the matrix $K$ is given by

$$K = dg(1, 1, 1, e_1).$$

By (2.3) and (2.13), the transformation

$$Y = UZ$$

takes (2.1) into

$$Z' = (\Lambda - U^{-1}U')Z.$$

Now by (2.13),

$$U^{-1}U' = K^{-1}T^{-1}T'K + K^{-1}K',$$

where

$$K^{-1}K' = dg(0, 0, 0, e_1^{-1}e_1'),$$

and we use (2.15).

Now we write

$$U^{-1}U' = \phi_{ij} \quad (1 \leq i, j \leq 4),$$

and

$$T^{-1}T' = \phi_{ij} \quad (1 \leq i, j \leq 4),$$

then by (2.18) to (2.21), we have

$$\phi_{ij} = \psi_{ij}, \quad (1 \leq i, j \leq 3),$$ (2.22)

$$\phi_{44} = \psi_{44} + e_1^{-1}e_1',$$ (2.23)

$$\phi_{i4} = \psi_{i4}e_1 \quad (1 \leq i \leq 3),$$ (2.24)

$$\phi_{j} = e_1^{-1}\psi_{j4} \quad (1 \leq j \leq 3).$$ (2.25)

Now to work out $\phi_{ij}(1 \leq i, j \leq 4)$, it suffices to deal with $\psi_{ij}$ of the matrix $T^{-1}T'$. Thus by (2.6), (2.10), (2.11) and (2.12) we obtain

$$\psi_{ii} = \frac{1}{2} m_i^2 \quad (1 \leq i \leq 4),$$

and, for $i \neq j, 1 \leq i, j \leq 4$

$$\psi_{ij} = m_i^{-1} \left\{ \lambda_i \left( p_0 \lambda_i^2 + \frac{1}{2} q_1 \lambda_i \right) + \lambda_i \left( p_0 \lambda_j^2 + \frac{1}{2} q_1 \lambda_j \right)' - \frac{1}{2} q_2' - (p_2 \lambda_j^{-1})' \right\}.$$ (2.27)

Now we need to work out (2.26) and (2.27) in some detail in terms of $p_0, p_1, p_2, q_1$ and $q_2$ and then (2.22)-(2.25) in order to determine the form of (2.17).

3. **THE MATRICES $L, T^{-1}T$ AND $U^{-1}U$**

In our analysis, we impose a basic condition on the coefficients, as follows:

(I) $p_i(0 \leq i \leq 2)$ and $q_i(i = 1, 2)$ are nowhere zero in some interval $[a, \infty)$, and
\[ \frac{p_i}{q_{i+1}} = o \left( \frac{q_{i+1}}{p_{i+1}} \right) \quad (i = 0, 1) \quad (x \to \infty) \]  

and

\[ \frac{q_i}{p_i} = o \left( \frac{p_i}{q_i} \right). \]  

If we write

\[ \epsilon_1 = \frac{p_0 p_1}{q_1^2}, \quad \epsilon_2 = \frac{q_1 q_2}{p_1^2}, \quad \epsilon_3 = \frac{p_2 p_1}{q_2^2}, \]  

then by (3.1) and (3.2) for \( 1 \leq i \leq 3 \)

\[ \epsilon_i = o(1) \quad (x \to \infty). \]  

Now as in [4], we can solve the characteristic equation (2.4) asymptotically as \( x \to \infty \). Using (3.1), (3.2) and (3.3) we obtain the distinct eigenvalues \( \lambda_j \) as

\[ \lambda_1 = -\frac{p_0}{q_1} (1 + \delta_1), \]  

\[ \lambda_2 = -\frac{q_1}{p_1} (1 + \delta_2), \]  

\[ \lambda_3 = -\frac{p_1}{q_1} (1 + \delta_3), \]  

and

\[ \lambda_4 = -\frac{q_1}{p_0} (1 + \delta_4), \]  

where

\[ \delta_1 = o(e_3), \quad \delta_2 = o(e_2) + o(e_3), \quad \delta_3 = o(e_1) + o(e_3), \quad \delta_4 = o(1). \]  

Now by (3.1) and (3.2), the ordering of \( \lambda_j \) is such that

\[ \lambda_j = o(\lambda_{j+1}) \quad (x \to \infty, 1 \leq j \leq 3). \]  

Now we work out \( m_j (1 \leq j \leq 4) \) asymptotically as \( x \to \infty \), hence by (3.3)-(3.9), (2.12) gives for \( 1 \leq j \leq 4 \)

\[ m_1 = q_2 \{ 1 + o(e_3) \}, \]  

\[ m_2 = -q_2 \{ 1 + o(e_2) + o(e_3) \}, \]  

\[ m_3 = \frac{p_1^2}{q_1} \{ 1 + o(e_1) + o(e_2) \}, \]  

and

\[ m_4 = -\frac{q_1^3}{p_0^2} \{ 1 + o(e_1) \}. \]  

Also on substituting \( \lambda_j (j = 1, 2, 3, 4) \) into (2.12) and using (3.5)-(3.8) respectively and differentiating, we obtain

\[ m_1' = q_2 \{ 1 + o(e_3) \} + q_2 \{ o(e_3) + o(e_2) \} + o(e_2^2 e_3^2) + o(e_1 e_2 e_3^2), \]  

\[ m_2' = -q_2 \{ 1 + o(e_2) + o(e_3) \} + o(e_2^2) + o(e_1 e_2 e_3^2). \]
\[ m'_2 = - q'_1 \{ 1 + o(\varepsilon_2) + o(\varepsilon_3) \} + q_2 \{ 0(\varepsilon'_2) + o(\varepsilon'_2) + o(\varepsilon'_1 \varepsilon'_2) \}, \]  (3.16)

\[ m'_3 = \left( \frac{p^3_{q_1}}{q_0} \right)' \{ 1 + o(\varepsilon_1) + o(\varepsilon_2) \} + \frac{p^3_{q_1}}{q_0} \{ 0(\varepsilon'_2) + o(\varepsilon'_2) + o(\varepsilon'_1) \}, \]  (3.17)

and

\[ m'_4 = - \left( \frac{q^3_{q_1}}{p_0^3} \right)' \{ 1 + o(\varepsilon_2) \} \left( \frac{q^3_{q_1}}{p_0^3} \right)' \{ 0(\varepsilon'_2 \varepsilon'_3) + o(\varepsilon'_1) \}. \]  (3.18)

At this stage we also require the following conditions

\[ \frac{p^i_{q_i}}{p_0} \varepsilon_i, \frac{p^i_{q_i}}{q_i} \varepsilon_i, \frac{q^i_{q_i}}{q_i} \varepsilon_i, \frac{p^i_{q_i}}{p_i} \varepsilon_i, \frac{p^i_{q_i}}{p_i} \varepsilon_i, \frac{p^i_{q_i}}{p_i} \varepsilon_i \text{ are all} \]

\[ L(a, \infty) \quad (1 \leq i \leq 3). \]  (3.19)

Further, differentiating (3.3) for \( \varepsilon_i (1 \leq i \leq 3) \), we obtain

\[ \varepsilon'_1 = 0 \left( \frac{p^i_{q_i}}{p_0} \varepsilon_1 \right) + 0 \left( \frac{p^i_{q_i}}{p_i} \varepsilon_1 \right) + o \left( \frac{q^i_{q_i}}{q_i} \varepsilon_1 \right), \]  (3.20)

\[ \varepsilon'_2 = 0 \left( \frac{q^i_{q_i}}{q_i} \varepsilon_2 \right) + 0 \left( \frac{q^i_{q_i}}{q_i} \varepsilon_2 \right) + o \left( \frac{p^i_{q_i}}{p_i} \varepsilon_2 \right), \]  (3.21)

and

\[ \varepsilon'_3 = 0 \left( \frac{p^i_{q_i}}{p_i} \varepsilon_3 \right) + 0 \left( \frac{p^i_{q_i}}{p_i} \varepsilon_3 \right) + o \left( \frac{q^i_{q_i}}{q_i} \varepsilon_3 \right). \]  (3.22)

For reference shortly, we note on substituting (3.5)-(3.8) into (2.4) and differentiating, we obtain

\[ \delta'_1 = 0(\varepsilon'_2) + 0(\varepsilon'_3 \varepsilon'_2) + o(\varepsilon'_1 \varepsilon'_3 \varepsilon'_2), \]  (3.23)

\[ \delta'_2 = 0(\varepsilon'_2) + 0(\varepsilon'_3) + o(\varepsilon'_1 \varepsilon'_3), \]  (3.24)

\[ \delta'_3 = 0(\varepsilon'_1) + 0(\varepsilon'_2) + o(\varepsilon'_1 \varepsilon'_3), \]  (3.25)

and

\[ \delta'_4 = 0(\varepsilon'_1) + 0(\varepsilon'_2 \varepsilon'_1) + o(\varepsilon'_3 \varepsilon'_1 \varepsilon'_2). \]  (3.26)

Hence by (3.19) and (3.20)-(3.26)

\[ \varepsilon'_j \text{ and } \delta'_j \text{ are } L(a, \infty). \]  (3.27)

For the diagonal elements \( \psi_{ii} (1 \leq j \leq 4) \) in (2.26) we can now substitute the estimates (3.11)-(3.18) into (2.26). We obtain

\[ \psi_{11} = \frac{1}{2} \left( \frac{q^i_{q_i}}{q_i} \varepsilon_3 \right) + o \left( \frac{q^i_{q_i}}{q_i} \varepsilon_3 \right) + 0(\varepsilon'_2) + o(\varepsilon'_3 \delta'_1) + o(\varepsilon'_3 \varepsilon'_2) + o(\varepsilon'_1 \varepsilon'_2), \]  (3.28)

\[ \psi_{22} = \frac{1}{2} \left( \frac{q^i_{q_i}}{q_i} \varepsilon_2 \right) + o \left( \frac{q^i_{q_i}}{q_i} \varepsilon_2 \right) + 0(\varepsilon'_2) + o(\varepsilon'_3 \delta'_2) + o(\varepsilon'_2) + o(\varepsilon'_1 \varepsilon'_2), \]  (3.29)
\[ \psi_{33} = \frac{1}{2} \left[ \frac{p_1}{p_1} - \frac{q_1}{q_1} \right] + \frac{q_1}{q_1} \left( \frac{p_1}{p_1} \right) + \frac{q_1}{q_1} \left( \frac{p_1}{p_1} \right) + O(\varepsilon_2^3) + O(\varepsilon_2^4) + O(\varepsilon_1). \]  
\[ \psi_{44} = \frac{1}{2} \left[ 3 \frac{q_1}{q_1} - 2 \frac{p_0}{p_0} \right] + \frac{q_1}{q_1} \left( \frac{p_0}{p_0} \right) + \frac{q_1}{q_1} \left( \frac{p_0}{p_0} \right) + O(\varepsilon_2^3) + O(\varepsilon_2^4) + O(\varepsilon_1). \]  

Now for the non-diagonal elements \( \psi_{ij} (i \neq j, 1 \leq i, j \leq 4) \), we consider (2.27). Hence (2.27) gives for \( i = 1 \) and \( j = 2 \)

\[ \psi_{12} = m_1^{-1} \left\{ \lambda_2 \left( p_0 \lambda_1^2 + \frac{1}{2} q_1 \lambda_1 \right) + \lambda_1 \left( p_0 \lambda_2^2 + \frac{1}{2} q_1 \lambda_2 \right) - \frac{1}{2} q_2^2 - (p_2 \lambda_2^{-1})' \right\}. \]  

Now by (3.5), (3.6), (3.3) and (3.11) we have

\[ m_1^{-1} \lambda_2 \left( p_0 \lambda_1^2 + \frac{1}{2} q_1 \lambda_1 \right) = \frac{1}{2} \left[ \frac{q_2}{q_2} - \frac{p_1}{p_1} \right] \varepsilon_2 \varepsilon_3 \left( 1 + O(\varepsilon_3) \right) + O(\varepsilon_2 \varepsilon_3 \varepsilon_2'), \]  

\[ m_1^{-1} \lambda_1 \left( p_0 \lambda_2^2 + \frac{1}{2} q_1 \lambda_2 \right)' = 0(\varepsilon_2 \varepsilon_3 \varepsilon_2') + 0(\varepsilon_2 \varepsilon_3 \varepsilon_2') \left[ \frac{p_0}{p_0} + 2 \frac{q_2}{q_2} - 2 \frac{p_1}{p_1} \right] \]  
\[ + 0(\varepsilon_2 \varepsilon_3) \frac{q_1}{q_1} + \frac{q_2}{q_2} - \frac{p_1}{p_1}, \]  

\[ - \frac{1}{2} q_2^2 \frac{1}{q_2} = - \frac{1}{2} q_2^2 + 0(\varepsilon_2 \varepsilon_3) \]  

and

\[ m_1^{-1} (p_2 \lambda_2^{-1})' = 0 \left( \frac{p_2}{p_2} \varepsilon_3 \right) + 0 \left( \frac{p_1}{p_1} \varepsilon_3 \right) + 0 \left( \frac{q_2}{q_2} \varepsilon_3 \right) + 0(\varepsilon_3 \varepsilon_2). \]  

Hence by (3.33)-(3.36), (3.32) gives

\[ \psi_{12} = - \frac{1}{2} \frac{q_2}{q_2} + 0 \left( \frac{q_2}{q_2} \varepsilon_3 \right) + 0 \left( \frac{p_1}{p_1} \varepsilon_3 \right) + 0 \left( \frac{p_1}{p_1} \varepsilon_3 \right) + 0 \left( \frac{p_0}{p_0} \varepsilon_3 \varepsilon_3 \right) + 0(\varepsilon_2 \varepsilon_3). \]  

Similar work can be done for the other elements \( \psi_{ij} \), so we obtain

\[ \psi_{13} = - \frac{1}{2} \frac{q_2}{q_2} + 0 \left( \frac{q_2}{q_2} \varepsilon_3 \right) + 0 \left( \frac{p_1}{p_1} \varepsilon_3 \right) + 0 \left( \frac{q_1}{q_1} \varepsilon_3 \right) + 0(\varepsilon_3 \varepsilon_2). \]  

\[ \psi_{14} = - \frac{1}{2} \frac{q_2}{q_2} + 0 \left( \frac{q_2}{q_2} \varepsilon_3 \right) + 0 \left( \frac{q_1}{q_1} \varepsilon_1^{-1} \varepsilon_3 \right) + 0 \left( \frac{p_0}{p_0} \varepsilon_1^{-1} \varepsilon_3 \right) + 0(\varepsilon_1 \varepsilon_3 \varepsilon_2). \]  

\[ \psi_{21} = - \frac{1}{2} \frac{q_2}{q_2} + 0 \left( \frac{q_2}{q_2} \varepsilon_2 \right) + 0 \left( \frac{q_2}{q_2} \varepsilon_2 \right) + 0(\varepsilon_2 \varepsilon_1 \varepsilon_2 \varepsilon_3) + 0 \left( \frac{p_0}{p_0} \varepsilon_1 \varepsilon_2 \varepsilon_3 \right). \]
\[
\psi_{23} = \left[ \frac{q_1}{2q_1} p_1 + \frac{q_2}{2q_2} \right] + 0\left( \frac{q_1}{q_1} \epsilon_1 \right) + 0\left( \frac{q_1}{q_1} \epsilon_2 \right) + 0\left( \frac{q_1}{q_1} \epsilon_3 \right) \\
+ 0\left( \frac{p_1}{p_1} \epsilon_1 \right) + 0\left( \frac{p_1}{p_1} \epsilon_2 \right) + 0\left( \frac{p_1}{p_1} \epsilon_3 \right) + 0\left( \frac{q_2}{q_2} \right) \\
+ 0(\delta_3) + 0\left( \frac{p_6}{p_0} \epsilon_1 \right) + 0\left( \epsilon_2 \epsilon_3 \frac{p_6}{p_0} \right). \tag{3.41}
\]

\[
\psi_{24} = \epsilon_1^{-1}\left[ \frac{q_1}{2q_1} + 0\left( \frac{q_1}{q_1} \epsilon_1 \right) + 0\left( \frac{q_1}{q_1} \epsilon_2 \right) + 0\left( \frac{q_1}{q_1} \epsilon_3 \right) + 0\left( \frac{\delta_3}{p_0} \epsilon_1 \right) \\
+ 0\left( \frac{p_6}{p_0} \epsilon_2 \right) + 0\left( \frac{p_6}{p_0} \epsilon_3 \right) + 0(\delta_4) + 0\left( \frac{\delta_4}{q_2} \epsilon_1 \right) + 0\left( \frac{p_6}{p_2} \epsilon_1^2 \epsilon_2 \epsilon_3 \right) \right]. \tag{3.42}
\]

\[
\psi_{31} = 0\left( \frac{p_6}{p_2} \epsilon_2 \right) + 0\left( \frac{q_2}{q_2} \epsilon_2 \right) + 0(\delta_1 \epsilon_2) + 0\left( \frac{q_1}{q_1} \epsilon_2 \epsilon_3 \right) + 0\left( \frac{p_6}{p_0} \epsilon_1 \epsilon_2^2 \epsilon_3 \right). \tag{3.43}
\]

\[
\psi_{32} = 0\left( \frac{q_2}{q_2} \epsilon_2 \right) + 0\left( \frac{p_1}{p_1} \epsilon_2 \right) + 0(\epsilon_2 \delta_2) + 0\left( \epsilon_1 \epsilon_2 \frac{p_6}{p_0} \right) + 0\left( \frac{q_1}{q_1} \epsilon_2 \right) + 0\left( \epsilon_2 \epsilon_3 \frac{p_6}{p_2} \right). \tag{3.44}
\]

\[
\psi_{34} = \epsilon_1^{-1}\left[ - \frac{q_1}{2q_1} + 0\left( \frac{q_1}{q_1} \epsilon_1 \right) + 0\left( \frac{q_1}{q_1} \epsilon_2 \right) + 0\left( \frac{p_6}{p_0} \epsilon_1 \right) + 0\left( \frac{p_6}{p_0} \epsilon_2 \right) \\
+ 0(\delta_4) + 0\left( \frac{q_1}{q_1} \epsilon_1 \epsilon_2 \right) + 0\left( \frac{p_6}{p_2} \epsilon_1^2 \epsilon_2 \epsilon_3 \right) \right]. \tag{3.45}
\]

\[
\psi_{41} = \epsilon_1 \left[ 0\left( \frac{q_1}{q_1} \epsilon_2 \epsilon_3 \right) + 0\left( \frac{q_2}{q_2} \epsilon_1 \epsilon_2 \right) + 0\left( \frac{p_6}{p_2} \epsilon_1 \epsilon_2 \right) + 0(\delta_1 \epsilon_1 \epsilon_2) + 0\left( \frac{p_6}{p_0} \epsilon_1 \epsilon_2^2 \epsilon_3 \right) \right]. \tag{3.46}
\]

\[
\psi_{42} = 0\left( \frac{q_2}{q_2} \epsilon_1 \epsilon_2 \right) + 0\left( \frac{p_1}{p_1} \epsilon_1 \epsilon_2 \right) + 0(\delta_2 \epsilon_1 \epsilon_2) + 0\left( \frac{q_1}{q_1} \epsilon_1 \epsilon_2 \right) \\
+ 0\left( \frac{p_6}{p_0} \epsilon_1 \epsilon_2 \right) + 0\left( \frac{p_6}{p_2} \epsilon_1 \epsilon_2 \epsilon_3 \right). \tag{3.47}
\]

\[
\psi_{43} = \epsilon_1 \left[ - \frac{q_1}{2q_1} + 0\left( \frac{p_1}{p_1} \epsilon_1 \right) + 0\left( \frac{q_1}{q_1} \epsilon_1 \right) + 0\left( \frac{q_1}{q_1} \epsilon_2 \right) + 0(\delta_3 \epsilon_1) \\
0\left( \frac{p_6}{p_0} \epsilon_1 \right) + 0\left( \frac{p_6}{p_2} \epsilon_1 \epsilon_2 \epsilon_3 \right) + 0\left( \frac{q_2}{q_2} \epsilon_1 \epsilon_2 \right) \right]. \tag{3.48}
\]

Now we need to work out (2.22)-(2.25) in order to determine the form (2.17). Now by (3.28)-(3.31) and (3.37)-(3.48), (2.22)-(2.25) will give:

\[
\phi_{11} = \frac{1}{2} q_2 + 0(\Delta_1), \quad \phi_{22} = \frac{1}{2} q_2 + 0(\Delta_2) \\
\phi_{33} = \frac{p_1}{p_1} - \frac{1}{2} q_2 + 0(\Delta_3), \quad \phi_{44} = \frac{p_1}{p_1} - \frac{1}{2} q_1 + 0(\Delta_4) \tag{3.49}
\]
\[ \phi_{12} = -\frac{1}{2} q_2^2 + O(\Delta_5), \quad \phi_{13} = -\frac{1}{2} q_2^2 + O(\Delta_6) \]
\[ \phi_{14} = 0(\Delta_7), \quad \phi_{21} = -\frac{1}{2} q_2^2 + O(\Delta_8) \]
\[ \phi_{23} = \frac{1}{2} \left( \frac{q_1^2}{q_1} + \frac{q_2^2}{q_2} \right) - \frac{p_1}{p_1} + O(\Delta_9), \quad \phi_{24} = \frac{1}{2} \frac{q_1^2}{q_1} + O(\Delta_{10}) \]
\[ \phi_{31} = 0(\Delta_{11}), \quad \phi_{32} = 0(\Delta_{12}), \quad \phi_{34} = -\frac{1}{2} \frac{q_1^2}{q_1} + O(\Delta_{13}) \]
\[ \phi_{41} = 0(\Delta_{14}), \quad \phi_{42} = 0(\Delta_{15}), \quad \phi_{43} = -\frac{1}{2} \frac{q_1^2}{q_1} + O(\Delta_{16}). \]

where

\[ \Delta_i \text{ is } L(a, \infty) \quad (1 \leq i \leq 16) \]

by (3.19) and (3.27).

Now by (3.49)-(3.51), we write the system (2.17) as

\[ Z' = (\Lambda + R + S)Z \]

where

\[ R = \begin{bmatrix}
-\eta_1 & \eta_1 & \eta_1 & 0 \\
\eta_1 & -\eta_1 & \eta_2 - \eta_1 & -\eta_3 \\
0 & 0 & -\eta_2 & \eta_3 \\
0 & 0 & \eta_3 & -\eta_2
\end{bmatrix} \]

with

\[ \eta_1 = \frac{1}{2} q_2^2, \quad \eta_2 = \frac{(p_1 q_1^{-1/2})'}{p_1 q_1^{-1/2}}, \quad \eta_3 = \frac{1}{2} q_2^2, \]

and \( S \) is \( L(a, \infty) \) by (3.51).

4. THE ASYMPTOTIC FORM OF SOLUTIONS

**THEOREM 4.1.** Let the coefficients \( q_1, q_2 \) and \( p_1 \) in (1.1) be \( C^{(2)}[a, \infty) \) and let \( p_0 \) and \( p_2 \) to be \( C^{(1)}[a, \infty) \). Let (3.1), (3.2) and (3.19) hold. Let

\[ \eta_k = \omega_k \frac{p_2}{q_2} (1 + \psi_k) \]

where \( \omega_k (1 \leq k \leq 3) \) are "non-zero" constants and \( \psi_k(x) \to 0 (1 \leq k \leq 3, x \to \infty) \). Also let

\[ \psi_k(x) \text{ is } L(a, \infty) \quad (1 \leq k \leq 3). \]

Let

\[ \text{Re } I_j(x) (j = 1, 2) \quad \text{and} \quad \text{Re} \left[ \frac{1}{2} (\lambda_3 + \lambda_4 + \eta_2 + \eta_4 - \lambda_1 - \lambda_2) \pm I_1 \pm I_2 \right] \]

be of one sign in \( [a, \infty) \)

where

\[ I_1 = \left[ 4 \eta_1^2 + (\lambda_1 - \lambda_2)^2 \right]^{1/2}, \]
\[ I_2 = \left[ 4 \eta_3^2 + (\lambda_3 - \lambda_4)^2 \right]^{1/2}. \]

Then (1.1) has solutions
\[ y_k \sim q_k^{-1/2} \exp \left( \frac{1}{2} \int_a^x \left[ \lambda_1 + \lambda_2 + (-1)^{k+1}I_1 \right] dt \right), \quad (k = 1, 2) \]

\[ y_3 \sim q_1^{1/2} p_1^{-1} \exp \left( \frac{1}{2} \int_a^x \left[ \lambda_3 + \lambda_4 + I_2 \right] dt \right), \]

\[ y_4 = o \left( q_1^{1/2} p_1^{-1} \exp \left( \frac{1}{2} \int_a^x \left[ \lambda_3 + \lambda_4 - I_2 \right] dt \right) \right). \]

**Proof.** As in [4] we apply Eastham Theorem [6, section 2] to the system (3.52) provided only that \( \Lambda \) and \( R \) satisfy the conditions and we shall use (3.53), (3.54), (4.1) and (4.2). We first require that

\[ \eta_k = o(\lambda_i - \lambda_j) \quad (i \neq j, 1 \leq i, k, j, \leq 4, k \neq 3), \]

this being [6, (2.1)] for our system. By (4.1), (3.54), (3.5)-(3.8), this requirement is implied by (3.1) and (3.2).

We also require that

\[ \{ \eta_k(\lambda_i - \lambda_j) \} \in L(\alpha, \infty) \quad (1 \leq k \leq 3), \]

for \( (i \neq j) \) this being [9, (2.2)] for our system. By (4.1), (3.54), (3.5)-(3.8), this requirement is implied by (3.19) and (4.2). Finally we require the eigenvalues \( \mu_k(1 \leq k \leq 4) \) of \( \Lambda + R \) satisfy the dichotomy condition [10], as in [4], the dichotomy condition holds if

\[ \mu_j - \mu_k = f + g(j \neq k, 1 \leq j, k \leq 4) \]

where \( f \) has one sign in \([\alpha, \infty)\) and \( g \in L(\alpha, \infty) \) [6, (1.5)]. Now by (2.3) and (3.53)

\[ \mu_k = \frac{1}{2}(\lambda_1 + \lambda_2 - 2\eta_1) + \frac{1}{2}(-1)^{k+1}I_1, \quad (k = 1, 2) \]

\[ \mu_k = \frac{1}{2}(\lambda_3 + \lambda_4 - 2\eta_2) + \frac{1}{2}(-1)^{k+1}I_2, \quad (k = 3, 4). \]

Thus by (4.3), (4.11) holds since (3.52) satisfies all the conditions for the asymptotic result [6, section 2], it follows that as \( x \to \infty \), (2.17) has four linearly independent solutions,

\[ Z_k(x) = \{e_k + o(1)\} \exp \left( \int_a^x \mu_k(t) dt \right), \]

where \( e_k \) is the coordinate vector with \( k \)-th component unity and other components zero. We now transform back to \( Y \) by means of (2.13) and (2.16). By taking the first component on each side of (2.16) and making use of (4.12) and (4.13) and carrying out the integration of \( -\frac{1}{2} m \) and \( \frac{q_i^m p_i}{q_i^m p_i} \) for \( (1 \leq k \leq 4) \) respectively we obtain (4.6), (4.7) and (4.8) after an adjustment of a constant multiple in \( y_k(1 \leq k \leq 3) \).

5. **Discussion**

(i) In the familiar case the coefficients which are covered by Theorem 4.1 are

\[ p_i(x) = c_i x^{\alpha_i}(i = 0, 1, 2), \quad q_i(x) = c_{i+2} x^{\alpha_{i+2}}(i = 1, 2) \]

with real constants \( \alpha_i \) and \( c_i(0 \leq i \leq 4) \). Then the critical case (4.1) is given by

\[ \alpha_4 - \alpha_2 = 1. \]

The values of \( \omega_k(1 \leq k \leq 3) \) in (4.1) are given by

...
\[ \omega_1 = \frac{1}{2} \alpha_4 c_2 c_4^{-1}, \quad \omega_2 = \left( \alpha_1 - \frac{1}{2} \alpha_3 \right) c_2 c_4^{-1}, \quad \omega_3 = \frac{1}{2} \alpha_3 c_2 c_4^{-1}, \tag{5.2} \]

where

\[ \psi_k(x) = 0 \quad (1 \leq k \leq 4). \tag{5.3} \]

(ii) More general coefficients are

\[ p_0 = c_0 x^{\alpha_0} e^{-2x}, \quad p_1 = c_1 x^{\alpha_1} e^{x}, \quad p_2 = c_2 x^{\alpha_2} e^{x}, \]
\[ q_1 = c_3 x^{\alpha_3} e^{-x}, \quad q_2 = c_4 x^{\alpha_4} e^{x}. \]

with real constants \( c_i, \alpha_i (0 \leq i \leq 4) \) and \( b > 0 \). Then the critical case (4.1) is given by

\[ \alpha_2 - \alpha_4 = b - 1 \tag{5.4} \]

and the values of \( \omega_k (1 \leq k \leq 4) \) are given by

\[ \omega_1 = \frac{1}{2} \frac{b c_4 c_2^{-1}}{c_2}, \quad \omega_2 = \frac{3}{2} \omega_1, \quad \omega_3 = -\frac{1}{2} \omega_1, \]

with \( \psi_1 = \alpha_4 b^{-1} x^{-b}, \psi_2 = \frac{4}{3} b^{-1} (\alpha_1 - \frac{1}{2} \alpha_3) x^{-b}, \psi_3 = -2 \alpha_3 b^{-1} x^{-b} \). Here it is clear that \( \psi_k \in L(\alpha, \infty) \) because \( b > 0 \).

(iii) We note that in both critical cases (5.1) and (5.4) represent an equation of line in the \( \alpha_2 \alpha_4 \)-plane.

REFERENCES
