ON GENERALIZATIONS OF THE POMPEIU FUNCTIONAL EQUATION

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(Received October 25, 1995 and in revised form January 15, 1997)

ABSTRACT. In this paper, we determine the general solution of the functional equations

\[ f(x + y + xy) = p(x) + q(y) + g(x)h(y), \quad (\forall x, y \in \mathbb{R}_+) \]

and

\[ f(ax + by + cxy) = f(x) + f(y) + f(x)f(y), \quad (\forall x, y \in \mathbb{R}) \]

which are generalizations of a functional equation studied by Pompeiu. We present a method which is simple and direct to determine the general solutions of the above equations without any regularity assumptions.

KEY WORDS AND PHRASES: Pompeiu functional equation, multiplicative function, logarithmic function, exponential function.


1. INTRODUCTION

Let \( \mathbb{R} \) be the set of all real numbers and \( \mathbb{R}_+ \) denote the set of nonzero reals. Further, let \( \mathbb{R}_+ = \mathbb{R} \setminus \{-1\} \), that is the set of real numbers except negative one. A function \( M : D \to \mathbb{R} \) is said to be multiplicative if and only if \( M(xy) = M(x)M(y) \) for all \( x, y \in D \), where \( D = \mathbb{R} \) or \( \mathbb{R}_+ \). A function \( E : \mathbb{R} \to \mathbb{R} \) is called exponential if and only if \( E(x + y) = E(x)E(y) \) for all \( x, y \in \mathbb{R} \). A function \( L : \mathbb{R}_+ \to \mathbb{R} \) is said to be logarithmic if and only if \( L(xy) = L(x) + L(y) \) for all \( x, y \in \mathbb{R}_+ \). A comprehensive treatment of these functions can be found in the book of Aczel and Dhombres [1].

If \( G = \mathbb{R}_+ \), then \( (G, \circ) \) is an abelian group where the group operation is defined as

\[ x \circ y = x + y + xy. \]

A characterization of the homomorphisms of the group \( (G, \circ) \) can be obtained by solving the functional equation

\[ f(x + y + xy) = f(x) + f(y) + f(x)f(y). \quad (PE) \]

This functional equation is known as the Pompeiu functional equation [3,4].
Suppose that \( f : \mathbb{R} \to \mathbb{R} \) satisfies (PE). Then the only solution \( f \) of the Pompeiu equation (PE) is given by
\[
f(x) = M(x + 1) - 1, \tag{1.1}
\]
where \( M \) is multiplicative.

To see this, add 1 to both sides of (PE) and write \( F(x) = 1 + f(x) \). Then (PE) reduces to \( F(x + y + xy) = F(x)F(y) \). Now replacing \( x \) by \( x - 1 \) and \( y \) by \( y - 1 \), we obtain \( M(xy) = M(x)M(y) \), where \( M(x) = F(x - 1) \). Thus, \( M \) is multiplicative and \( f(x) = F(x) - 1 = M(x + 1) - 1 \), which is (1.1).

In a special case, \( f \) is an automorphism of the field \( \mathbb{R} \). Suppose \( M \) is also additive. Then \( M \) is a ring homomorphism of \( \mathbb{R} \). If \( M \) is a nontrivial homomorphism, then \( f(x) = M(x) = x \), that is, \( f \) is an automorphism of the field \( \mathbb{R} \).

In this paper, we determine the general solution of the functional equations
\[
f(x + y + xy) = p(x) + q(y) + g(x)h(y), \quad (\forall x, y \in \mathbb{R}_*) \tag{FE1}
\]
and
\[
f(ax + by + cxy) = f(x) + f(y) + f(x)f(y), \quad (\forall x, y \in \mathbb{R}) \tag{FE2}
\]
which are generalizations of the Pompeiu functional equation (PE). We present a method which is simple and direct to determine the general solutions of (FE1) and (FE2) without any regularity assumptions. For other related functional equations, the interested reader should refer to [2] and [5].

2. SOME PRELIMINARY RESULTS

The following two lemmas will be instrumental for establishing the main result of this paper.

**LEMMA 1.** Let \( g, h : \mathbb{R}_o \to \mathbb{R} \) satisfy the functional equation
\[
g(xy) = g(y) + g(x)h(y) \tag{2.1}
\]
for all \( x, y \in \mathbb{R}_o \). Then for all \( x, y \in \mathbb{R}_o \), \( g(x) \) and \( h(y) \) are given by
\[
g(x) = 0, \quad h(y) = \text{arbitrary}; \tag{2.2}
\]
\[
g(x) = L(x), \quad h(y) = 1; \tag{2.3}
\]
\[
g(x) = \alpha [M(x) - 1], \quad h(y) = M(y), \tag{2.4}
\]
where \( M : \mathbb{R}_o \to \mathbb{R} \) is a multiplicative map not identically one, \( L : \mathbb{R}_o \to \mathbb{R} \) is a logarithmic function not identically zero and \( \alpha \) is an arbitrary nonzero constant.

**PROOF.** If \( g \equiv 0 \), then \( h \) is arbitrary and they satisfy the equation (2.1). Hence we have the solution (2.2). We assume hereafter that \( g \not\equiv 0 \).

Interchanging \( x \) with \( y \) in (2.1) and comparing the resulting equation to (2.1), we get
\[
g(y)[h(x) - 1] = g(x)[h(y) - 1]. \tag{2.5}
\]
Suppose \( h(x) = 1 \) for all \( x \in \mathbb{R}_o \). Then (2.1) yields \( g(xy) = g(y) + g(x) \) and hence the function \( g : \mathbb{R}_o \to \mathbb{R} \) is logarithmic. This yields the solution (2.3).
Finally, suppose \( h(y) \neq 1 \) for some \( y \). Then from (2.5), we have

\[
g(x) = \alpha [h(x) - 1],
\]

(2.6)

where \( \alpha \) is a nonzero constant, since \( g \not= 0 \). Using (2.6) in (2.1), and simplifying, we obtain

\[
h(xy) = h(x) h(y).
\]

(2.7)

Hence, \( h : \mathbb{R} \to \mathbb{R} \) is a multiplicative function. This gives the asserted solution (2.4) and the proof of the lemma is now complete.

**Lemma 2.** The general solutions \( f, g, h : \mathbb{R} \to \mathbb{R} \) of the functional equation

\[
f(xy) = f(x) + f(y) + \alpha g(x) + \beta h(y) + g(x)h(y) \quad (\forall x, y \in \mathbb{R})
\]

(2.8)

where \( \alpha \) and \( \beta \) are apriori chosen constants, have values \( f(x), g(x) \) and \( h(y) \) given, for all \( x, y \in \mathbb{R} \), by

\[
\begin{align*}
f(x) &= L(x) + \alpha \beta \\
g(x) &= c L_1(x) - \beta \\
h(y) &= L(y) - \alpha;
\end{align*}
\]

(2.9)

\[
\begin{align*}
f(x) &= L_0(x) + L_1(x) + \alpha \beta \\
g(x) &= c L_1(x) - \beta \\
h(y) &= L(y) - \alpha
\end{align*}
\]

(2.10)

\[
\begin{align*}
f(x) &= L(x) + \gamma [M(x) - 1] + \alpha \beta \\
g(x) &= \gamma [M(x) - 1] - \beta \\
h(y) &= \delta [M(y) - 1] - \alpha,
\end{align*}
\]

(2.11)

(2.12)

where \( M : \mathbb{R} \to \mathbb{R} \) is a multiplicative map not identically one, \( L_0, L_1, L : \mathbb{R} \to \mathbb{R} \) are logarithmic functions with \( L_1 \) not identically zero, and \( c, \delta, \gamma \) are arbitrary nonzero constants.

**Proof.** Interchanging \( x \) with \( y \) in (2.8) and comparing the resulting equation to (2.8), we obtain

\[
[\alpha + h(y)][\beta + g(x)] = [\alpha + h(x)][\beta + g(y)].
\]

(2.13)

Now we consider several cases.

**Case 1.** Suppose \( h(y) = -\alpha \) for all \( y \in \mathbb{R} \). Then (2.8) yields

\[
f(xy) = f(x) + f(y) - \alpha \beta.
\]

(2.14)

Hence

\[
f(x) = L(x) + \alpha \beta.
\]

(2.15)

where \( L : \mathbb{R} \to \mathbb{R} \) is a logarithmic function. Hence we have the asserted solution (2.9).

**Case 2.** Suppose \( g(x) = -\beta \) for all \( x \in \mathbb{R} \). Then (2.8) yields

\[
f(xy) = f(x) + f(y) - \alpha \beta.
\]
Hence, as before,

\[ f(x) = L(x) + \alpha \beta, \]

where \( L : \mathbb{R} \to \mathbb{R} \) is a logarithmic function. Thus we have the asserted solution (2.10).

**Case 3.** Now we assume \( h(x) \neq -\alpha \) for some \( x \in \mathbb{R}_o \) and \( g(x) \neq -\beta \) for some \( x \in \mathbb{R}_o \). From (2.13), we get

\[ \beta + g(y) = c [\alpha + h(y)], \tag{2.16} \]

where \( c \) is a nonzero constant.

Using (2.8), we compute

\[
\begin{align*}
f(x \cdot yz) &= f(x) + f(y) + f(z) + \alpha g(y) + \beta h(z) \\
&\quad + g(y)h(z) + \alpha g(x) + \beta h(yz) + g(x)h(yz).
\end{align*}
\]

Again, using (2.8), we have

\[
\begin{align*}
f(xy \cdot z) &= f(x) + f(y) + f(z) + \alpha g(x) + \beta h(y) \\
&\quad + g(x)h(y) + \alpha g(xy) + \beta h(z) + g(xy)h(z).
\end{align*}
\]

From (2.17) and (2.18), we obtain

\[
[a + h(z)] [g(y) - g(xy)] = [\beta + g(x)] [h(y) - h(yz)], \quad \forall x, y \in \mathbb{R}_o. \tag{2.19}
\]

Since \( g(x) \neq -\beta \) for some \( x \in \mathbb{R}_o \), there exists a \( x_o \in \mathbb{R}_o \) such that \( g(x_o) + \beta \neq 0 \). Letting \( x = x_o \) in (2.19), we have

\[ h(yz) = h(y) + [\alpha + h(z)] k(y), \tag{2.20} \]

where

\[ k(y) = \frac{g(yx_o) - g(y)}{g(x_o) + \beta}. \tag{2.21} \]

The general solution of (2.20) can be obtained from Lemma 1 (add \( \alpha \) to both sides). Hence, taking into consideration that \( h(y) + \alpha \neq 0 \), we have

\[ h(y) = L_1(y) - \alpha. \tag{2.22} \]

or

\[ h(y) = \delta [M(y) - 1] - \alpha, \tag{2.23} \]

where \( L_1 \) is logarithmic not identically zero, \( M \) is multiplicative not identically one, and \( \delta \) is an arbitrary constant.

Now we consider two subcases.

**Subcase 3.1.** From (2.22) and (2.16), we have

\[ g(y) = c L_1(y) - \beta. \tag{2.24} \]

Using (2.22) and (2.24) in (2.8), we get

\[ f(xy) = f(x) + f(y) + c L_1(x) L_1(y) - \alpha \beta. \tag{2.25} \]

Defining

\[ L_0(x) := f(x) - \frac{1}{2} c L_1^2(x) - \alpha \beta, \tag{2.26} \]
we see that (2.25) reduces to
\[ L_o(xy) = L_o(x) + L_o(y) \]
for all \( x, y \in \mathbb{R}_o \), that is, \( L_o \) is logarithmic and from (2.26), we have
\[ f(x) = L_o(x) + \frac{1}{2} c L^2_o(x) + \alpha \beta. \]  
(2.27)
Hence (2.27), (2.24) and (2.22) yield the asserted solution (2.11).

**Subcase 3.2.** Finally, from (2.23) and (2.16), we obtain
\[ g(y) = \delta c [M(y) - 1] - \beta. \]  
(2.28)
With (2.23) and (2.28) in (2.8), we have
\[ f(xy) = f(x) + f(y) - \alpha \beta + c \delta^2 [M(x) - 1][M(y) - 1]. \]  
(2.29)
Defining
\[ L(x) := f(x) - c \delta^2 [M(x) - 1] - \alpha \beta, \]  
(2.30)
we see that (2.29) reduces to
\[ L(xy) = L(x) + L(y) \]
for all \( x, y \in \mathbb{R}_o \), that is, \( L \) is a logarithmic function. Using (2.30), we have
\[ f(x) = L(x) + \gamma \delta [M(x) - 1] + \alpha \beta, \]  
(2.31)
where \( \gamma = c \delta \). Hence (2.31), (2.28) and (2.23) yield the asserted solution (2.12). This completes the proof of the lemma.

### 3. SOLUTION OF THE FUNCTIONAL EQUATION (FE1)

Now we are ready to present the general solution of (FE1) using Lemma 2.

**THEOREM 1.** The functions \( f, p, q, g, h : \mathbb{R}_+ \to \mathbb{R} \) satisfy the functional equation
\[ f(x + y + xy) = p(x) + q(y) + g(x) h(y) \]  
(FE1)
for all \( x, y \in \mathbb{R}_+ \) if and only if, for all \( x, y \in \mathbb{R}_+ \),
\[
\begin{align*}
  f(x) &= L(x + 1) + \alpha \beta + a + b \\
p(x) &= L(x + 1) + b \\
  q(y) &= L(y + 1) + \alpha \beta + a + \beta h(y) \\
  g(x) &= -\beta \\
  h(y) &= \text{arbitrary}; \\
\end{align*}
\]  
(3.1)
and
\[
\begin{align*}
  f(x) &= L(x + 1) + \alpha \beta + a + b \\
p(x) &= L(x + 1) + a + \alpha \beta + b + \alpha g(x) \\
  q(y) &= L(y + 1) + a \\
  g(x) &= \text{arbitrary} \\
  h(y) &= -\alpha; \\
\end{align*}
\]  
(3.2)
\begin{align*}
  f(x) &= L(x + 1) + \gamma [M(x + 1) - 1] + \alpha \beta + a + b \\
  p(x) &= L(x + 1) + (\delta + \alpha) \gamma [M(x + 1) - 1] + b \\
  q(y) &= L(y + 1) + (\gamma + \beta) \delta [M(y + 1) - 1] + a \\
  g(x) &= \gamma [M(x + 1) - 1] - \beta \\
  h(y) &= \delta [M(y + 1) - 1] - \alpha;
\end{align*}

(3.3)

where \( M : \mathbb{R}_o \to \mathbb{R} \) is a multiplicative function not identically one, \( L_o, L_1, L : \mathbb{R}_o \to \mathbb{R} \) are logarithmic maps with \( L_1 \) not identically zero, and \( \alpha, \beta, \gamma, \delta, a, b, c \) are arbitrary real constants.

**Proof.** First, we substitute \( y = 0 \) in (FE1) and then we put \( x = 0 \) in (FE1) to obtain

\[ p(x) = f(x) - a + \alpha g(x) \]

(3.5)

and

\[ q(y) = f(y) - b + \beta h(y), \]

(3.6)

where \( a := q(0), b := p(0), \alpha := -h(0), \beta := -g(0). \) Using (3.5) and (3.6) in (FE1), we have

\[ f(x + y + xy) = f(x) + f(y) + f(x)f(y), \quad x, y \in \mathbb{R}_o \]

(3.7)

for \( x, y \in \mathbb{R}_o \). Replacing \( x \) by \( u - 1 \) and \( y \) by \( v - 1 \) in (3.7) and then defining

\[ F(u) := f(u - 1) - a - b, \quad G(u) := g(u - 1), \quad H(u) := h(u - 1) \]

(3.8)

for all \( u \in \mathbb{R}_o \), we obtain

\[ F(uv) = F(u) + F(v) + \alpha G(u) + \beta H(v) + G(u)H(v) \]

(3.9)

for all \( u, v \in \mathbb{R}_o \). The general solution of (3.9) can now be obtained from Lemma 2. The first two solutions of Lemma 2 (see (2.9) and (2.10)) together with (3.5) and (3.6) yield the solutions (3.1) and (3.2). The next two solutions of Lemma 2 (that is, solution (2.11) and (2.12)) yield together with (3.5) and (3.6) the asserted solutions (3.3) and (3.4). This completes the proof of the theorem.

**4. Solution of the Functional Equation (FE2)**

Let \( a, b \) and \( c \) be real parameters. We consider the functional equation

\[ f(ax + by + cxy) = f(x) + f(y) + f(x)f(y), \quad \forall x, y \in \mathbb{R}. \]

(FE2)

The only constant solutions of (FE2) are \( f \equiv 0 \) and \( f \equiv -1 \). So we look for nonconstant solutions of the functional equation (FE2).

Substitution of \( x = 0 = y \) in (FE2) yields \( f(0)[f(0) + 1] = 0 \). Hence, either \( f(0) = 0 \) or \( f(0) = -1 \). Now we consider two cases.
Case 1. Suppose \( f(0) = -1 \). Then \( x = 0 \) in (FE2) gives \( f(by) = f(0) \), so that when \( b \neq 0 \), \( f \) is a constant which is not the case. Similarly by putting \( y = 0 \) in (FE2), we get \( f \) is a constant when \( a \neq 0 \).

Suppose \( a = 0 = b \). If \( c \) is also zero, then (FE2) is \( [1 + f(x)] [1 + f(y)] = 0 \) since \( f(0) = -1 \). That is \( f \) is a constant. So, assume \( c \neq 0 \). Then replacing \( x \) by \( \frac{x}{c} \) and \( y \) by \( \frac{y}{c} \) in (FE2), we obtain

\[
M(xy) = M(x) M(y),
\]

(4.1)

where \( M : \mathbb{R} \to \mathbb{R} \) is a multiplicative map with \( M(x) = 1 + f\left(\frac{x}{c}\right) \). Hence

\[
f(x) = M(cx) - 1
\]

(4.2)

is a solution of (FE2) with \( f(0) = -1, a = 0 = b, c \neq 0 \).

Case 2. Suppose \( f(0) = 0 \). Let \( a = 0 \). Then \( y = 0 \) in (FE2) gives \( f = 0 \) which is not the case. So, \( a \neq 0 \). Similarly \( b \neq 0 \). Setting \( x = 0 \) and \( y = 0 \) separately in (FE2), we get

\[
f(by) = f(y) \quad \text{and} \quad f(ax) = f(x)
\]

(4.3)

so that (FE2) becomes

\[
f(ax + by + cxy) = f(ax) + f(by) + f(ax) f(by).
\]

(4.4)

Suppose \( c = 0 \). Then replacing \( x \) by \( \frac{x}{a} \) and \( y \) by \( \frac{y}{b} \) in (4.4) we have

\[
E(x + y) = E(x) E(y)
\]

where \( E : \mathbb{R} \to \mathbb{R} \) given by

\[
E(x) = 1 + f(x)
\]

(4.5)

is an exponential map. Further, from (4.3) and (4.5), we get

\[
E(ax) = E(x) = E(bx)
\]

and since \( E(x) E(-x) = 1 \), so we get

\[
E((a - b)x) = 1 = E((a - 1)x).
\]

(4.6)

If \( a \neq b \), then \( E \) is a constant map and so \( f \) is also a constant function. If \( a \neq 1 \), then \( E \) and so \( f \) is a constant. Hence \( a = 1 = b \). Thus by (4.5)

\[
f(x) = E(x) - 1
\]

is a solution of (FE2) with \( a = b = 1, c = 0 \).

Finally, let \( a \neq 0, b \neq 0 \) and \( c \neq 0 \). Set \( \alpha = \frac{x}{a} \). Replacing \( x \) by \( \frac{x}{a} \) and \( y \) by \( \frac{y}{b} \) in (4.4), we obtain

\[
F(x + y + xy) = F(x) F(y),
\]

(4.7)

where

\[
F(x) = 1 + f\left(\frac{x}{\alpha}\right).
\]

(4.8)

Changing \( x \) to \( x - 1 \) and \( y \) to \( y - 1 \) in (4.7) we have

\[
M(xy) = M(x) M(y),
\]
where $M : \mathbb{R} \to \mathbb{R}$ is multiplicative and

$$M(x) = F(x - 1).$$

Thus by (4.8) and (4.9), we have

$$f(x) = F(ax - 1) = M(1 + ax - 1).$$

(4.10)

If we use (4.10) in (4.3), and recall that $\alpha = \frac{c}{a}$, we get

$$M \left(1 + \frac{c}{a} x\right) = M \left(1 + \frac{x}{c} \right) = M \left(1 + \frac{x}{a} \right).$$

(4.11)

Recall that, since $M$ is multiplicative, $M(x)M \left(\frac{1}{b}\right) = 1$ (otherwise if $M(1) = 0$, then $M \equiv 0$ so that $f \equiv -1$). Changing separately $x$ to $\frac{ax}{c}$ and $x$ to $\frac{bx}{c}$ in (4.11), we obtain

$$M(1 + x) = M \left(1 + \frac{x}{c} \right) = M \left(1 + \frac{x}{a} \right).$$

(4.12)

Similarly, replacing $x$ by $\frac{ax}{c}$ in (4.11), we have

$$M(1 + x) = M(1 + ax) = M(1 + bx).$$

(4.13)

Replacing $x$ by $x - 1$ in (4.13), we obtain $M(x) = M(1 + a(x - 1))$ which yields

$$M \left(\frac{1-a+ax}{x}\right) = 1 \quad \text{if } x \neq 0.$$  

Suppose $a \neq 1$. Changing $x$ to $(1 - a)x$, we have $M(1 + \frac{x}{a}) = 1$ and thus (again replacing $x$ by $\frac{1}{x-a}$) we have $M(x) = 1$ when $x \neq a$. Similarly, if $b \neq 1$, we get $M(x) = 1$ when $x \neq 0, b$.

Hence, $M(x) = 1$ for all $x$ which leads to $f$ is a constant. Therefore $a = 1 = b$. Then from (4.10), we obtain

$$f(x) = M(1 + cx) - 1$$

(4.14)

where $M : \mathbb{R} \to \mathbb{R}$ is multiplicative. Thus we have proved the following theorem.

THEOREM 2. The function $f : \mathbb{R} \to \mathbb{R}$ is a solution of (FE2) if and only if $f(x)$, for every $x \in \mathbb{R}$, is given by

$$f(x) = \begin{cases} 
M(cx) - 1 & \text{if } a = 0 = b, c \neq 0 \\
E(x) - 1 & \text{if } a = 1 = b, c = 0 \\
M(cx + 1) - 1 & \text{if } a = 1 = b, c \neq 0 \\
k & \text{otherwise,}
\end{cases}$$

where $M : \mathbb{R} \to \mathbb{R}$ is multiplicative, $E : \mathbb{R} \to \mathbb{R}$ is exponential, and $k$ is a constant satisfying $k(k + 1) = 0$.

ACKNOWLEDGMENTS. We are thankful to the referee for suggestions that improved the presentation of this paper. This research is partially supported by a grant from the Graduate Programs and Research of the University of Louisville.

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