THE ABEL-TYPE TRANSFORMATIONS INTO $\ell$

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ABSTRACT. Let $t$ be a sequence in $(0,1)$ that converges to 1, and define the Abel-type matrix $A_{\alpha,t}$ by $a_{nk} = \binom{k+\alpha}{k} t_n^{k+1} (1-t_n)^{\alpha+1}$ for $\alpha > -1$. The matrix $A_{\alpha,t}$ determines a sequence-to-sequence variant of the Abel-type power series method of summability introduced by Borwein in [1]. The purpose of this paper is to study these matrices as mappings into $\ell$. Necessary and sufficient conditions for $A_{\alpha,t}$ to be $\ell$-$\ell$, $G$-$\ell$, and $G_w$-$\ell$ are established. Also, the strength of $A_{\alpha,t}$ in the $\ell$-$\ell$ setting is investigated.

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1. Introduction and background. The Abel-type power series method [1], denoted by $A_{\alpha}$, $\alpha > -1$, is the following sequence-to-function transformation: if

$$\sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k < \infty \quad \text{for } 0 < x < 1$$

(1.1)

and

$$\lim_{x \to 1^-} (1-x)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k = L,$$

(1.2)

then we say that $u$ is $A_{\alpha}$-summable to $L$. In order to study this summability method as a mapping into $\ell$, we must modify it into a sequence to sequence transformation. This is achieved by replacing the continuous parameter $x$ with a sequence $t$ such that $0 < t_n < 1$ for all $n$ and $\lim t_n = 1$. Thus, the sequence $u$ is transformed into the sequence $A_{\alpha,t} u$ whose $n$th term is given by

$$(A_{\alpha,t} u)_n = (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k t_n^k.$$  

(1.3)

This transformation is determined by the matrix $A_{\alpha,t}$ whose $nk$th entry is given by

$$a_{nk} = \binom{k+\alpha}{k} t_n^k (1-t_n)^{\alpha+1}.$$  

(1.4)

The matrix $A_{\alpha,t}$ is called the Abel-type matrix. The case $\alpha = 0$ is the Abel matrix introduced by Fridy in [5]. It is easy to see that the $A_{\alpha,t}$ matrix is regular and, indeed, totally regular.
2. Basic notations. Let $A = (a_{nk})$ be an infinite matrix defining a sequence-to-sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,$$  \hspace{1cm} (2.1)

where $(Ax)_n$ denotes the $n$th term of the image sequence $Ax$. The sequence $Ax$ is called the $A$-transform of the sequence $x$. If $X$ and $Z$ are sets of complex number sequence, then the matrix $A$ is called an $X$-$Z$ matrix if the image $Au$ of $u$ under the transformation $A$ is in $Z$ whenever $u$ is in $X$.

Let $y$ be a complex number sequence. Throughout this paper, we use the following basic notations:

$$\ell = \{ y : \sum_{k=0}^{\infty} |y_k| \text{ converges} \},$$

$$\ell^p = \{ y : \sum_{k=0}^{\infty} |y_k|^p \text{ converges} \},$$

$$d(A) = \{ y : \sum_{k=0}^{\infty} a_{nk} y_k \text{ converges for each } n \geq 0 \},$$

$$\ell(A) = \{ y : Ay \in \ell \},$$

$$G = \{ y : y_k = O(r^k) \text{ for some } r \in (0,1) \},$$

$$G_w = \{ y : y_k = O(r^k) \text{ for some } r \in (0,w), 0 < w < 1 \},$$

$$c(A) = \{ y : y \text{ is summable by } A \}. $$

3. The main results. Our first result gives a necessary and sufficient condition for $A_{\alpha,t}$ to be $\ell$-$\ell$.

**Theorem 1.** Suppose that $-1 < \alpha \leq 0$. Then the matrix $A_{\alpha,t}$ is $\ell$-$\ell$ if and only if $(1-t)^{\alpha+1} \in \ell$.

**Proof.** Since $-1 < \alpha \leq 0$ and $0 < t_n < 1$, we have

$$\sum_{n=0}^{\infty} |a_{nk}| = \left( k + \alpha \right) \sum_{n=0}^{\infty} t_n^k (1-t_n)^{\alpha+1} \leq \sum_{n=0}^{\infty} (1-t_n)^{\alpha+1} \text{ for each } k.$$  \hspace{1cm} (3.1)

Thus, if $(1-t)^{\alpha+1} \in \ell$, Knopp-Lorentz theorem [6] guarantees that $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix. Also, if $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then by Knopp-Lorentz theorem, we have

$$\sum_{n=0}^{\infty} |a_{n,o}| < \infty.$$  \hspace{1cm} (3.2)

and this yields $(1-t)^{\alpha+1} \in \ell$. \hfill $\Box$

**Remark 1.** In Theorem 1, the implication that $A_{\alpha,t}$ is $\ell$-$\ell$ $\Rightarrow$ $(1-t)^{\alpha+1} \in \ell$ is also true for any $\alpha > 0$, however, the converse implication is not true for any $\alpha > 0$. This is demonstrated in Theorem 4 below.
Corollary 1. If $-1 < \alpha \leq 0$ and $0 < t_n < w_n < 1$, then $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix whenever $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix.

Proof. The corollary follows easily by Theorem 1.

Corollary 2. If $-1 < \alpha < \beta \leq 0$, then $A_{\beta,t}$ is an $\ell$-$\ell$ matrix whenever $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix.

Corollary 3. If $-1 < \alpha \leq 0$ and $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then $1/\log(1-t) \in \ell$.

Corollary 4. If $-1 < \alpha \leq 0$, then $\arcsin((1-t)^{\alpha+1}) \in \ell$ if and only if $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix.

Corollary 5. Suppose that $-1 < \alpha \leq 0$ and $w_n = 1/t_n$. Then the zeta matrix $z_w$ is in $\ell$-$\ell$ whenever $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix.

Corollary 6. Suppose that $-1 < \alpha \leq 0$ and $t_n = 1 - (n+2)^{-q}, 0 < q < 1$; then $A_{\alpha,t}$ is not an $\ell$-$\ell$ matrix.

Proof. Since $(1-t)^{\alpha+1}$ is not in $\ell$, the corollary follows easily by Theorem 1.

Before considering our next theorem, we recall the following result which follows as a consequence of the familiar Hölder’s inequality for summation. The result states that if $x$ and $y$ are real number sequences such that $x \in \ell^p, y \in \ell^q, p > 1$, and $(1/p) + (1/q) = 1$, then $x y \in \ell$.

Theorem 2. If $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then

$$\sum_{n=0}^{\infty} \frac{\log(2-t_n)}{(n+1)} < \infty. \tag{3.3}$$

Proof. Since $\log(2-t_n) \sim (1-t_n)$, it suffices to show that

$$\sum_{n=0}^{\infty} \frac{(1-t_n)}{(n+1)} < \infty. \tag{3.4}$$

If $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then, by Theorem 1, we have $(1-t)^{\alpha+1} \in \ell$. If $-1 < \alpha \leq 0$, it is easy to see that if $(1-t)^{\alpha+1} \in \ell$, then we have $(1-t) \in \ell$ and, consequently, the assertion follows. If $\alpha > 0$, then the theorem follows using the preceding result by letting $x_n = 1 - t_n, y_n = 1/(n+1), p = \alpha + 1,$ and $q = (\alpha+1)/\alpha$.

Theorem 3. Suppose that $t_n = (n+1)/(n+2)$. Then $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix if and only if $\alpha > 0$.

Proof. If $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then, by Theorem 1, it follows that $(1-t)^{\alpha+1} \in \ell$ and this yields $\alpha > 0$. Conversely, suppose that $\alpha > 0$. Then we have

$$\sum_{n=0}^{\infty} |a_{nk}| = \binom{k+\alpha}{k} \sum_{n=0}^{\infty} \frac{n+1}{n+2} \binom{\alpha+1}{n+2} x^n \leq M \binom{k+\alpha}{k} \int_{0}^{\infty} (x+1)^{k} (x+2)^{-(k+\alpha+1)} dx \tag{3.5}$$
for some $M > 0$. This is possible as both the summation and the integral are finite since $\alpha > 0$. Now, we let

$$g(k) = \int_0^\infty (x + 1)^k (x + 2)^{-(k+\alpha+1)} \, dx,$$

(3.6)
and we compute $g(k)$ using integration by parts repeatedly. We have

$$g(k) = \frac{1}{k + \alpha} \cdot 2^{-(k+\alpha)} + h_1(k),$$

(3.7)

where

$$h_1(k) = \frac{k}{k + \alpha} \int_0^\infty (x + 1)^{k-1} (x + 2)^{-(k+\alpha)} \, dx$$

$$= \frac{k \cdot 2^{-(k+\alpha-1)}}{(k + \alpha)(k + \alpha - 1)} + h_2k$$

(3.8)

and

$$h_2(k) = \frac{k(k-1)}{(k + \alpha)(k + \alpha - 1)} \int_0^\infty (x + 1)^{k-2} (x + 2)^{-(k+\alpha-1)} \, dx$$

$$= \frac{k(k-1) \cdot 2^{-(k+\alpha-2)}}{(k + \alpha)(k + \alpha - 1)(k + \alpha - 2)} + h_3(k).$$

(3.9)

It follows that

$$h_3(k) = \frac{k(k-1)(k-2) \cdot 2^{-(k+\alpha-3)}}{(k + \alpha)(k + \alpha - 1)(k + \alpha - 2)(k + \alpha - 3)} + h_4(k),$$

(3.10)

where

$$h_4(k) = \frac{k(k-1)(k-2)(k-3)}{(k + \alpha)(k + \alpha - 1)(k + \alpha - 2)(k + \alpha - 3)(k + \alpha - 4)}$$

$$\times \int_0^\infty (x + 1)^{k-4} (x + 2)^{-(k+\alpha-3)} \, dx.$$  

(3.11)

Continuing this process, we get

$$h_k(k) = \frac{k(k-1)(k-2) \cdots 2^{-(k-\alpha)}}{(k + \alpha)(k + \alpha - 1)(k + \alpha - 2) \cdots \alpha} = \frac{2^{-\alpha}}{\alpha(k + \alpha)^k}.$$  

(3.12)

It is easy to see that $g(k)$ can be written using summation notation as

$$g(k) = \frac{2^{-\alpha}}{\alpha(k + \alpha)} \sum_{i=0}^k \binom{i + \alpha - 1}{i} 2^{-i}$$

$$\leq \frac{2^{-\alpha}}{\alpha(k + \alpha)} \sum_{i=0}^\infty \binom{i + \alpha - 1}{i} 2^{-i}$$

$$= \frac{2^{-\alpha}}{\alpha(k + \alpha)} 2^\alpha = \frac{1}{\alpha(k + \alpha)}.$$  

(3.13)
Consequently, we get
\[
\sum_{n=0}^{\infty} |a_{nk}| \leq M \binom{k + \alpha}{k} g(k) \leq \frac{M(k + \alpha)}{\alpha(k + \alpha)} = \frac{M}{\alpha}.
\] (3.14)

Thus by the Knopp-Lorentz theorem [6], \( A_{\alpha,t} \) is an \( \ell \)-\( \ell \) matrix.

**Corollary 7.** Suppose \( t_n = \frac{n+1}{n+2} \). Then \( A_{\alpha,t} \) is an \( \ell \)-\( \ell \) matrix if and only if \((1 - t)^{\alpha + 1} \in \ell\).

**Theorem 4.** Suppose \( \alpha > 0 \) and \( t_n = 1 - (n+2)^{-q} \), \( 0 < q < 1 \). Then \( A_{\alpha,t} \) is not an \( \ell \)-\( \ell \) matrix.

**Proof.** If \((1 - t)^{\alpha + 1}\) is not in \( \ell \), then by Theorem 1, \( A_{\alpha,t} \) is not \( \ell \)-\( \ell \). If \((1 - t)^{\alpha + 1} \in \ell\), then we prove that \( A_{\alpha,t} \) is not \( \ell \)-\( \ell \) by showing that the condition of the Knopp-Lorentz theorem [6] fails to hold. For convenience, we let \( q = \frac{1}{p} \) and \( 2^{1/p} = R \), where \( p > 1 \). Then we have
\[
\sum_{n=0}^{\infty} |a_{nk}| = \binom{k + \alpha}{k} \sum_{n=0}^{\infty} (1 - (n+2)^{-1/p})(n+2)^{(-1/p)(\alpha + 1)}
= \binom{k + \alpha}{k} \sum_{n=0}^{\infty} ((n+2)^{1/p} - 1)^k(n+2)^{(-1/p)(k + \alpha + 1)}
\geq M \binom{k + \alpha}{k} \int_{0}^{\infty} ((x+2)^{1/p} - 1)^k(x+2)^{(-1/p)(k + \alpha + 1)} dx
\] (3.15)

for some \( M > 0 \). This is possible as both the summation and integral are finite since \((1 - t)^{\alpha + 1} \in \ell\). Now, let us define
\[
g(k) = \int_{0}^{\infty} ((x+2)^{1/p} - 1)^k(x+2)^{(-1/p)(k + \alpha + 1)} dx.
\] (3.16)

Using integration by parts repeatedly, we can easily deduce that
\[
g(k) = \frac{p(R-1)^kR^{-(\alpha + 1 - p)}}{k + \alpha + 1 - p} + \frac{pk(R-1)^{k-1}(R)^{-(\alpha + 1 - p)}}{(k + \alpha + 1 - p)(k + \alpha - p)}
+ \cdots + \frac{pk(k-1)(k-2) \cdots (R)^{-(\alpha + 1 - p)}}{(k + \alpha + 1 - p)(k + \alpha - p)(k + \alpha - 1 - p) \cdots (\alpha + 1 - p)}.
\] (3.17)

This implies that
\[
g(k) > \frac{pk(k-1)(k-2) \cdots R^{-(\alpha + 1 - p)}}{(k + \alpha + 1 - p)(k + \alpha - p)(k + \alpha - 1 - p) \cdots (\alpha + 1 - p)}
= \frac{pR^{-(\alpha + 1 - p)}}{(\alpha + 1 - p) \binom{k + \alpha + 1 - p}{k}}.
\] (3.18)
Now, we have
\[ \sum_{n=0}^{\infty} |a_{nk}| \geq M_1 \left( k + \alpha \right) \frac{g(k)}{k} \]
\[ pM_1 \left( k + \alpha \right) \frac{R^{-(\alpha+1-p)}}{(k+\alpha+1-p)^r} > \frac{M_2 k^\alpha}{k^{\alpha+1-p}} = M_2 k^{p-1}. \]  
(3.19)

Thus, it follows that
\[ \sup_k \left\{ \sum_{n=0}^{\infty} |a_{nk}| \right\} = \infty, \]  
(3.20)
and hence \( A_{\alpha,t} \) is not \( \ell_\ell \).

In case \( t_n = 1 - (n+2)^{-q} \), it is natural to ask whether \( A_{\alpha,t} \) is a \( \ell_\ell \) matrix. For \(-1 < \alpha \leq 0\), it is easy to see that \( A_{\alpha,t} \) is \( \ell_\ell \) if and only if \( \alpha > (1-q)/q \), by Theorem 1.

For \( \alpha > 0 \), the answer to this question is given by the next theorem, which gives a necessary and sufficient condition for the matrix to be \( \ell_\ell \).

**Theorem 5.** Suppose that \( \alpha > 0 \) and \( t_n = 1 - (n+2)^{-q} \). Then \( A_{\alpha,t} \) is an \( \ell_\ell \) matrix if and only if \( q \geq 1 \).

**Proof.** Suppose that \( q \geq 1 \). Let \( q = 1/p, 2^{1/p} = R \) and \((R-1)/R = S\), where \( 0 < p \leq 1 \). Then we have
\[ \sum_{n=0}^{\infty} |a_{nk}| = \left( k + \alpha \right) \sum_{n=0}^{\infty} \left( 1 - (n+2)^{-1/p} \right)^k \left( n+2 \right)^{(-1/p)(\alpha+1)} \]
\[ = \left( k + \alpha \right) \sum_{n=0}^{\infty} \left( n+2 \right)^{1/p} - 1 \right)^k \left( n+4 \right)^{(-1/p)(k+\alpha+1)} \]
\[ \leq M \left( k + \alpha \right) \int_0^{\infty} \left( (x+2)^{1/p} - 1 \right)^k (x+2)^{(-1/p)(k+\alpha+1)} \, dx \]
(3.21)

for some \( M > 0 \). This is possible as both the summation and the integral are finite since \((1-t)^{\alpha+1} \in \ell\) for \( \alpha > 0 \). Now, let us define
\[ g(k) = \int_0^{\infty} \left( (x+2)^{1/p} - 1 \right)^k (x+2)^{(-1/p)(k+\alpha+1)} \, dx. \]
(3.22)

Using integration by parts repeatedly, we can easily deduce that
\[ g(k) = \frac{p(R-1)^k R^{-(k+\alpha+1)}}{k+\alpha-p+1} + \frac{p k^k R^{-k-1}(R)^{-(k+\alpha)}}{(k+\alpha-p+1)(k+\alpha-p)} \]
\[ + \cdots + \frac{p k^{(k-1)} R^{-(k+\alpha)}}{(k+\alpha-p+1)(k+\alpha-p) \cdots (\alpha+p+1)}. \]
(3.23)

Now, from the hypotheses that \( q \geq 1 \) and \( \alpha > 0 \), it follows that
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$$g(k) \leq \frac{(R - 1)^{k+\alpha} R^{-(k+\alpha)}}{k+\alpha} + \frac{k(R - 1)^{k+\alpha-1} R^{-(k+\alpha-1)}}{(k+\alpha)(k+\alpha-1)}$$

$$+ \cdots + \frac{k(k-1)(k-2) \cdots R^{-(\alpha)}}{(k+\alpha)(k+\alpha-1) \cdots (\alpha)}$$

$$\leq \frac{S^{k+\alpha}}{k+\alpha} + \frac{kS^{k+\alpha-1}}{(k+\alpha)(k+\alpha-1)} + \cdots + \frac{k(k-1)(k-2) \cdots S^{\alpha}}{(k+\alpha)(k+\alpha-1) \cdots (\alpha)}.$$  (3.24)

By writing the right-hand side of the preceding inequality using the summation notation, we obtain

$$g(k) \leq \frac{S^\alpha}{\alpha} \sum_{i=0}^{k} \binom{i+\alpha-1}{i} S^i$$

$$\leq \frac{S^\alpha}{\alpha} \sum_{i=0}^{\infty} \binom{i+\alpha-1}{i} S^i$$

$$= \frac{S^\alpha}{\alpha} S^{-\alpha} = \frac{1}{\alpha(\alpha+1)}.$$  (3.25)

Consequently, we have

$$\sum_{n=0}^{\infty} |a_{nk}| \leq M \binom{k+\alpha}{k} g(k) \leq \frac{M \binom{k+\alpha}{k}}{\alpha(\alpha+1)} = \frac{M}{\alpha}.$$  (3.26)

Thus, by Knopp-Lorentz theorem [6], $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix.

Conversely, if $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then it follows, by Theorems 3 and 4, that $q \geq 1$.

**Corollary 8.** Suppose that $t_n = 1 - (n+2)^{-q}$, $w_n = 1 - (n+2)^{-p}$ and $q < p$. Then $A_{\alpha,w}$ is an $\ell$-$\ell$ matrix whenever $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix.

**Proof.** The result follows immediately from Theorems 1 and 5.

**Corollary 9.** Suppose that $\alpha > 0$, $t_n = 1 - (n+2)^{-q}$, $w_n = 1 - (n+2)^{-p}$ and $(1/q) + (1/p) = 1$. Then both $A_{\alpha,t}$ and $A_{\alpha,w}$ are $\ell$-$\ell$ matrices.

**Proof.** The hypotheses imply that both $q$ and $p$ are greater than 1, and hence the corollary follows easily by Theorem 5.

**Theorem 6.** The following statements are equivalent:

1. $A_{\alpha,t}$ is a $G_w$-$\ell$ matrix;
2. $(1-t)^{\alpha+1} \in \ell$;
3. arcsin$(1-t)^{\alpha+1} \in \ell$;
4. $((1-t)^{\alpha+1})/(\sqrt{1-(1-t)^{2(\alpha+1)}}) \in \ell$;
5. $A_{\alpha,t}$ is a $G$-$\ell$ matrix.
**Proof.** We get \((1) \Rightarrow (2)\) by [9, Thm. 1.1] and \((2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)\) follow easily from the following basic inequality

\[
x < \arcsin x < \frac{x}{\sqrt{1-x^2}}, \quad 0 < x < 1,
\]

and by [4, Thm. 1]. The assertion that \((5) \Rightarrow (1)\) follows immediately as \(G_w\) is a subset of \(G\).

**Corollary 10.** Suppose that \(t_n = 1 - (n + 2)^{-q}\). Then \(A_{\alpha,t}\) is a \(G\)-\(\ell\) matrix if and only if \(\alpha > (1 - q)/q\). For \(q = 1\), \(A_{\alpha,t}\) is a \(G\)-\(\ell\) matrix if and only if it is an \(\ell\)-\(\ell\) matrix.

**Proof.** The proof follows using Theorems 3 and 6.

**Theorem 7.** The following statements are equivalent:

1. \(A_{\alpha,t}\) is a \(G_w\)-\(G\) matrix;
2. \((1-t)^{\alpha+1} \in G_i\);
3. \(\arcsin(1-t)^{\alpha+1} \in G\);
4. \(A_{\alpha,t}\) is a \(G\)-\(G\) matrix.

**Proof.** \((1) \Rightarrow (2)\) follows by [9, Thm. 2.1] and \((2) \Rightarrow (3) \Rightarrow (4)\) follow easily from (3.27) and [4, Thm. 4]. The assertion that \((4) \Rightarrow (1)\) follows immediately as \(G_w\) is a subset of \(G\).

**Corollary 11.** If \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix, then it is a \(G\)-\(G\) matrix.

Our next few results suggest that the Abel-type matrix \(A_{\alpha,t}\) is \(\ell\)-stronger than the identity matrix (see [7, Def. 3]). The results indicate how large the sizes of \(\ell(A_{\alpha,t})\) and \(d(A_{\alpha,t})\) are.

**Theorem 8.** Suppose that \(-1 < \alpha \leq 0\), \(A_{\alpha,t}\) is an \(\ell\)-\(\ell\) matrix, and the series \(\sum_{k=0}^{\infty} x_k\) has bounded partial sums. Then it follows that \(x \in \ell(A_{\alpha,t})\).

**Proof.** Since, for \(-1 < \alpha \leq 0\), \((\frac{k+\alpha}{k})\) is decreasing, the theorem is proved by following the same steps used in the proof of [7, Thm. 4].

**Remark 2.** Although the preceding theorem is stated for \(-1 < \alpha \leq 0\), the conclusion is also true for \(\alpha > 0\) for some sequences. This is demonstrated as follows: let \(x\) be the bounded sequence given by

\[
x_k = (-1)^k.
\]

Let \(Y\) be the \(A_{\alpha,t}\)-transform of the sequence \(x\). Then it follows that the sequence \(Y\) is given by

\[
Y_n = (1 - t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^k = (1 - t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} (-1)^k t_n^k = \frac{(1-t_n)^{\alpha+1}}{(1+t_n)^{\alpha+1}}
\]
which implies that

$$Y_n < (1 - t_n)_{\alpha+1}^{\alpha+1}. \quad (3.30)$$

Hence, if $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then by Theorem 1, $(1 - t)^{\alpha+1} \in \ell$, and so $x \in \ell(A_{\alpha,t})$.

**Corollary 12.** Suppose that $-1 < \alpha \leq 0$, $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix. Then $\ell(A_{\alpha,t})$ contains the class of all sequences $x$ such that $\sum_{k=0}^{\infty} x_k$ is conditionally convergent.

**Remark 3.** In fact, we can give a further indication of the size of $\ell(A_{\alpha,t})$ by showing that if $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then it also contains an unbounded sequence. To verify this, consider the sequence $x$ given by

$$x_k = (-1)^k \frac{k + \alpha + 1}{\alpha + 1}. \quad (3.31)$$

Let $Y$ be the $A_{\alpha,t}$-transform of the sequence $x$. Then we have

$$Y_n = (1 - t_n)_{\alpha+1}^{\alpha+1} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^k$$

$$= (1 - t_n)_{\alpha+1}^{\alpha+1} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} (-1)^k \frac{k + \alpha + 1}{\alpha + 1} t_n^k$$

$$= \frac{(1 - t_n)_{\alpha+1}^{\alpha+1}}{(1 + t_n)_{\alpha+1}^{\alpha+1}}$$

and, consequently,

$$Y_n < (1 - t_n)_{\alpha+1}^{\alpha+1}. \quad (3.33)$$

Hence, if $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix, then by Theorem 1, $(1 - t)^{\alpha+1} \in \ell$, and so $x \in \ell(A_{\alpha,t})$.

This example clearly indicates that $A_{\alpha,t}$ is a rather strong method in the $\ell$-$\ell$ setting for any $\alpha > -1$.

The $\ell$-$\ell$ strength of the $A_{\alpha,t}$ matrices can also be demonstrated by comparing them with the familiar Norland matrices ($N_p$) [3]. By using the same techniques used in the proof of [3, Thm. 8], we can show that the class of the $A_{\alpha,t}$ matrix summability methods is $\ell$-stronger than the class of $N_p$ matrix summability methods for some $p$.

When discussing the $\ell$-$\ell$ strength of $A_{\alpha,t}$, or the size of $\ell(A_{\alpha,t})$, it is very important that we also determine the domain of $A_{\alpha,t}$. The following proposition, which can be easily proved, gives a characterization of the domain of $A_{\alpha,t}$.

**Proposition 1.** The complex number sequence $x$ is in the domain of the matrix $A_{\alpha,t}$ if and only if

$$\limsup_{k} |x_k|^{1/k} \leq 1. \quad (3.34)$$

**Remark 4.** Proposition 1 can be used as a powerful tool in making a comparison between the $\ell$-$\ell$ strength of the $A_{\alpha,t}$ matrices and some other matrices as shown by the following examples.

**Example 1.** The $A_{\alpha,t}$ matrix is not $\ell$-stronger than the Borel matrix $B[8, p. 53]$. To demonstrate this, consider the sequence $x$ given by

$$x_k = (-3)^k. \quad (3.35)$$
Then we have
\[(Bx)_n = \sum_{k=0}^{\infty} e^{-n} \frac{n^k}{k!} (-3)^k = e^{-4n}. \quad (3.36)\]
Thus, we have \(Bx \in \ell\) and hence \(x \in \ell(B)\), but by Proposition 1, \(x \notin \ell(A_{\alpha,t})\). Hence, \(A_{\alpha,t}\) is not \(\ell\)-stronger than \(B\).

**Example 2.** The \(A_{\alpha,t}\) matrix is not \(\ell\)-stronger than the familiar Euler-Knopp matrix \(E_r\) for \(r \in (0, 1)\). Also, \(E_r\) is not \(\ell\)-stronger than \(A_{\alpha,t}\). To demonstrate this, consider the sequence \(x\) defined by
\[x_k = (-q)^k \quad \text{and} \quad r = \frac{1}{q}, \quad (3.37)\]
where \(q > 1\). Let \(Y\) be the \(E_r\)-transform of the sequence \(x\). Then it is easy to see that the sequence \(Y\) is defined by
\[Y_n = \left(\frac{-1}{q}\right)^n. \quad (3.38)\]
Since \(q > 1\), we have \(Y \in \ell\) and hence \(x \in \ell(E_r)\), but \(x \notin \ell(A_{\alpha,t})\) by Proposition 1. Hence, \(A_{\alpha,t}\) is not \(\ell\)-stronger than \(E_r\). To show that \(E_r\) is not \(\ell\)-stronger than \(A_{\alpha,t}\), we let \(-1 < \alpha \leq 0\) and consider the sequence \(x\) that was constructed by Fridy in his example of [5, p. 424]. Here, we have \(x \notin \ell(E_r)\), but \(x \in \ell(A_{\alpha,t})\) by Theorem 8. Thus, \(E_r\) is not \(\ell\)-stronger than \(A_{\alpha,t}\).

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**References**


**Lemma:** Department of mathematics, Savannah State University, Savannah, Georgia 31404, USA