RINGS WITH MANY IDEMPOTENTS

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ABSTRACT. We introduce a new stable range condition and investigate the structures of rings with many idempotents. These are also generalizations of corresponding results of J. Stock and H. P. Yu.

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In this paper, we examine the properties of rings satisfying idempotent 1-stable range and give one large class of such rings. We show that many useful exchange rings belong to the new class of rings. As an application, we also give a new element-wise characterization of strongly π -regular rings. These are generalizations of many known results.

Throughout, R is an associative ring with identity. $M_n(R)$ denotes the ring of $n \times n$ matrices over R . Let $M_n(R)$ has an identity I_n , and let its group of units be the general linear group $GL_n(R)$. Set

$$\begin{aligned} B_{ij}(x) &= I_2 + x e_{ij} \quad (i \neq j, 1 \leq i, j \leq 2), \\ [\alpha, \beta] &= \alpha e_{11} + \beta e_{22}, \end{aligned} \tag{1}$$

where e_{11}, e_{22} and e_{ij} ($i \neq j, 1 \leq i, j \leq 2$) are all matrix units.

DEFINITION 1. A ring R is said to satisfy idempotent 1-stable range provided that for any $a, b \in R$, $aR + bR = R$ implies there exists an idempotent $e \in R$ such that $a + be$ is left invertible in R .

PROPOSITION 2. *The following are equivalent:*

- (1) *R satisfies idempotent 1-stable range.*
- (2) *For any $a, b \in R$, $aR + bR = R$ implies there exists an idempotent $e \in R$ such that $a + be \in U(R)$.*

PROOF. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) Given $aR + bR = R$. Then there exists an idempotent $e \in R$ such that $a + be = u$ is left invertible in R . Assume that $vu = 1$ for some $v \in R$. Then $vR + 0R = R$. Thus, we can find an idempotent $f \in R$ such that $v + 0 \cdot f = v$ is left invertible in R . So v is a unit, and then $a + be$ is a unit. $\square$

COROLLARY 3. *The following are equivalent:*

- (1) *R satisfies idempotent 1-stable range.*

- (2) For any $a, b \in R$, $aR + bR = R$ implies there exists an idempotent $e \in R$ such that $a + be$ is right invertible in R .

PROOF. (1) $\Rightarrow$ (2) is clear from Proposition 2.

(2) $\Rightarrow$ (1) Given $aR + bR = R$, then there exists an idempotent $e \in R$ such that $a + be = u$ is right invertible. Assume that $uv = 1$ for some $v \in R$. Since

$$vR + (1 - vu)R = R, \quad (2)$$

we can find an idempotent $f \in R$ such that

$$v + (1 - vu)f = w \quad (3)$$

is right invertible in R . Obviously,

$$uw = u(v + (1 - vu)f) = 1. \quad (4)$$

This implies that w is a unit. So $a + be$ is a unit, as required. $\square$

Now we investigate elements in 2-dimensional general linear groups over rings satisfying idempotent 1-stable range. As an application, we shall give an element-wise characterization of such rings.

THEOREM 4. *The following are equivalent:*

- (1) R satisfies idempotent 1-stable range.
 (2) For any $A \in \text{GL}_2(R)$, there exists an idempotent $e \in R$ such that

$$A = [* , *]B_{21}(*)B_{12}(*)B_{21}(-e). \quad (5)$$

PROOF. (1) $\Rightarrow$ (2) Given any $A = (a_{ij}) \in \text{GL}_2(R)$. Then we have

$$a_{11}R + a_{12}R = R. \quad (6)$$

So we can find an idempotent $f \in R$ such that

$$a_{11} + a_{12}f = u \in U(R). \quad (7)$$

It is easy to verify that

$$B_{21}(-(a_{21} + a_{22}f)u^{-1})AB_{21}(f)B_{12}(-u^{-1}a_{12}) = [u, a_{22} - (a_{21} + a_{22}f)u^{-1}a_{12}]. \quad (8)$$

So

$$A = [* , *]B_{21}(*)B_{12}(*)B_{21}(-f). \quad (9)$$

Let $e = -f$. Thus the result follows.

(2) $\Rightarrow$ (1) Given $aR + bR = R$. Then $ax + by = 1$ for some $x, y \in R$. It is easy to verify that

$$\begin{pmatrix} a & by \\ 1 & -x \end{pmatrix} = B_{12}(a) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_{12}(-x) \in \text{GL}_2(R). \quad (10)$$

So we can find an idempotent $e \in R$ such that

$$\begin{pmatrix} a & by \\ 1 & -x \end{pmatrix} = [* , *]B_{21}(*)B_{12}(*)B_{21}(-e). \quad (11)$$

Thus, $a + bye = u \in U(R)$. So we can verify the following.

$$\begin{pmatrix} a & b \\ ye & 1 \end{pmatrix} = B_{12}(b) \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} B_{21}(ye) \in GL_2(R). \quad (12)$$

Consequently, there is an idempotent $f \in R$ such that

$$\begin{pmatrix} a & b \\ ye & 1 \end{pmatrix} = [* , *]B_{21}(*)B_{12}(*)B_{21}(-f), \quad (13)$$

and then

$$\begin{pmatrix} a & b \\ ye & 1 \end{pmatrix} B_{21}(f) = [* , *]B_{21}(*)B_{12}(*). \quad (14)$$

Therefore $a + bf \in U(R)$, as desired. $\square$

THEOREM 5. *The following are equivalent:*

- (1) *R satisfies idempotent 1-stable range.*
- (2) *For any $x, y \in R$, there exists an idempotent $e \in R$ such that $xy - xe + 1 \in U(R)$.*

PROOF. (1) $\Rightarrow$ (2) For any $x, y \in R$,

$$(1 + xy)R + (-x)R = R. \quad (15)$$

So we can find an idempotent $e \in R$ such that

$$xy - xe + 1 = (1 + xy) + (-x)e \in U(R). \quad (16)$$

(2) $\Rightarrow$ (1) Given $xy + b = 1$, there exists an idempotent $e \in R$ such that

$$(-y)x - (-y)e + 1 \in U(R). \quad (17)$$

Let $x - e = a$. Then

$$1 - ya = u \in U(R). \quad (18)$$

Clearly, we have

$$x(1 - ya) - ba = x - (xy + b)a = x - a = e. \quad (19)$$

So

$$x - ba(1 - ya)^{-1} = e(1 - ya)^{-1}. \quad (20)$$

From $xy + b = 1$, we have

$$(x - ba(1 - ya)^{-1})y + b(1 + a(1 - ya)^{-1}y) = 1. \quad (21)$$

Hence,

$$e(1 - ya)^{-1}y + b(1 + a(1 - ya)^{-1}y) = 1. \quad (22)$$

So

$$e(1 - ya)^{-1}y(1 - e) + b(1 + a(1 - ya)^{-1}y)(1 - e) = 1 - e, \quad (23)$$

and then

$$e + b(1 + a(1 - ya)^{-1}y)(1 - e) = 1 - e(1 - ya)^{-1}y(1 - e). \quad (24)$$

Clearly,

$$1 - e(1 - ya)^{-1}y(1 - e) = (1 + e(1 - ya)^{-1}y(1 - e))^{-1} \in U(R). \quad (25)$$

So

$$\begin{aligned} x + b(-a(1 - ya)^{-1} + (1 + a(1 - ya)^{-1}y)(1 - e)(1 - ya)^{-1}) \\ = x - ba(1 - ya)^{-1} + b(1 + a(1 - ya)^{-1}y)(1 - e)(1 - ya)^{-1} \\ = e(1 - ya)^{-1} + b(1 + a(1 - ya)^{-1}y)(1 - e)(1 - ya)^{-1} \\ = (1 - e(1 - ya)^{-1}y(1 - e))(1 - ya)^{-1} \in U(R). \end{aligned} \quad (26)$$

Therefore R has stable range one.

Given any $A = (a_{ij}) \in \text{GL}_2(R)$, there are $h, k \in R$ such that

$$a_{11}h + a_{12}k = 1. \quad (27)$$

Since R has stable range one, there exists a $z \in R$ such that

$$a_{11} + a_{12}z = q \in U(R). \quad (28)$$

It is easy to verify that

$$B_{21}(-(a_{21} + a_{22}z)q^{-1})AB_{21}(z)B_{12}(-q^{-1}a_{12}) = [q, a_{22} - (a_{21} + a_{22}z)q^{-1}a_{12}]. \quad (29)$$

Obviously,

$$a_{22} - (a_{21} + a_{22}z)q^{-1}a_{12} \in U(R), \quad (30)$$

and then we have $m, n \in R$ such that

$$A = [* , *]B_{21}(*)B_{12}(m)B_{21}(n). \quad (31)$$

So there is an idempotent $f \in R$ such that

$$1 + m(n + 1 - f) = v \in U(R). \quad (32)$$

Let $e = 1 - f$ and $n = e + s$, then $n = -e$. Consequently, we see that

$$A = [* , *]B_{21}(*)B_{12}(m)B_{21}(s)B_{21}(-e). \quad (33)$$

Since $1 + ms \in U(R)$, one can verify

$$B_{12}(m)B_{21}(s) = [1 + ms, 1]B_{21}(s)B_{12}(m)\left[1, (1 + sm)^{-1}\right], \quad (34)$$

whence

$$A = [* , *]B_{21}(*)B_{12}(*)B_{21}(-e). \quad (35)$$

According to Theorem 4, we complete the proof. $\square$

As an immediately consequence, we now derive the following result which shows that idempotent 1-stable range property is left-right symmetric.

COROLLARY 6. *The following are equivalent:*

- (1) *R satisfies idempotent 1-stable range.*
- (2) *For any $a, b \in R$, $Ra + Rb = R$ implies there exists an idempotent $e \in R$ such that $a + eb \in U(R)$.*

PROOF. R satisfies idempotent 1-stable range if and only if for any $x, y \in R$, there exists an idempotent $e \in R$ such that

$$xy - xe + 1 = 1 + x(y - e) = u \in U(R). \quad (36)$$

Then a direct computation gives

$$\begin{aligned} (1 + (y - e)x)(1 - (y - e)(x + (u^{-1} - 1)x)) \\ = (1 - (y - e)(x + (u^{-1} - 1)x))(1 + (y - e)x) = 1, \end{aligned} \quad (37)$$

whence we can verify that

$$xy - xe + 1 = 1 + x(y - e) \in U(R) \quad (38)$$

if and only if

$$1 + (y - e)x \in U(R) \quad (39)$$

if and only if

$$x^0 y^0 - x^0 e^0 + 1^0 = 1^0 + x^0(y^0 - e^0) \in U(R^0). \quad (40)$$

Consequently, from Theorem 5, we see that R satisfies idempotent 1-stable range if and only if so does the opposite ring R^0 . Hence the result follows. $\square$

COROLLARY 7. *The following are equivalent:*

- (1) *R satisfies idempotent 1-stable range.*
- (2) *For any $A \in \text{GL}_2(R)$, there exists an idempotent $e \in R$ such that*

$$A = [* , *]B_{12}(*)B_{21}(*)B_{12}(e).$$

PROOF. Replacing A by its inverse A^{-1} , we know that condition (2) can be seen to be equivalent to the following condition: for any $A \in \text{GL}_2(R)$, there exists an idempotent $e \in R$ such that the transpose $A^t = B_{12}(-e)B_{21}(*)B_{12}(*)[* , *]$. In view of Theorem 4, we show that condition (2) is equivalent to the opposite ring R^0 satisfies idempotent 1-stable range. Using Corollary 6, we obtain the result. $\square$

COROLLARY 8. *The following are equivalent:*

- (1) *R satisfies idempotent 1-stable range.*
- (2) *Given $ax + b = 1$ in R . Then there exists an idempotent $e \in R$ such that $ae + b \in U(R)$.*
- (3) *Given $ax + b = 1$ in R . Then there exists an idempotent $e \in R$ such that $ex + b \in U(R)$.*

PROOF. (1) $\Rightarrow$ (2) Given $ax + b = 1$ in R . Then $bR + aR = R$. So there exists an idempotent $e \in R$ such that $ae + b \in U(R)$, as asserted.

(2) $\Rightarrow$ (1) For any $x, y \in R$, we have

$$(-x)y + (1 + xy) = 1. \quad (41)$$

So we can find an idempotent $e \in R$ such that

$$(-x)e + (1 + xy) \in U(R). \quad (42)$$

That is,

$$xy - xe + 1 \in U(R). \quad (43)$$

Therefore the result follows from Theorem 5.

(1) $\Leftrightarrow$ (3) is obvious by the left-right symmetry of idempotent 1-stable range condition. $\square$

THEOREM 9. *The following are equivalent:*

- (1) R satisfies idempotent 1-stable range.
- (2) $R/J(R)$ satisfies idempotent 1-stable range and idempotents can be lifted modulo $J(R)$.

PROOF. (1) $\Rightarrow$ (2) Given any $x + J(R), y + J(R) \in R/J(R)$. Since R satisfies idempotent 1-stable range, by virtue of Theorem 5, there is an idempotent $e \in R$ such that $xy - xe + 1 \in U(R)$. Thus we have

$$(x + J(R))(y + J(R)) - (x + J(R))(e + J(R)) + (1 + J(R)) \in U\left(\frac{R}{J(R)}\right) \quad (44)$$

with

$$e + J(R) = (e + J(R))^2 \in \frac{R}{J(R)}. \quad (45)$$

Using Theorem 5, we show that $R/J(R)$ satisfies idempotent 1-stable range.

Given any $a \in R$. We have $aR + (-1)R = R$. So there exists an idempotent $e \in R$ such that $a - e = u$, and then $a = e + u$. Thus R is a clean ring. By [11, Prop. 1.8, Thm. 1.1], R is exchange. Using [11, Cor. 1.3], we see that idempotents can be lifted modulo $J(R)$.

(2) $\Rightarrow$ (1) Given $aR + bR = R$. Then we have

$$(a + J(R))\left(\frac{R}{J(R)}\right) + (b + J(R))\left(\frac{R}{J(R)}\right) = \frac{R}{J(R)}. \quad (46)$$

Since $R/J(R)$ satisfies idempotent 1-stable range, there is an idempotent

$$e + J(R) \in \frac{R}{J(R)} \quad (47)$$

such that

$$(a + J(R)) + (b + J(R))(e + J(R)) \in U\left(\frac{R}{J(R)}\right). \quad (48)$$

As idempotents can be lifted modulo $J(R)$, we may assume $e = e^2 \in R$. On the other hand, there is some $v \in R$ such that

$$v(a + be) - 1 \in J(R). \quad (49)$$

Hence $a + be$ is left invertible, as desired. $\square$

EXAMPLE 10. Every local ring satisfies idempotent 1-stable range.

PROOF. Since R is local, $R/J(R)$ is a division ring. Let

$$S = \frac{R}{J(R)}. \quad (50)$$

Given $aS + bS = S$ with $a, b \in S$. If $a = 0$, then $bS = S$. So $a + b \cdot 1 = b$ is right invertible in S . If $a \neq 0$, then $a + b \cdot 0 = a$ is a unit in S . By virtue of Corollary 3, we show that $S = R/J(R)$ satisfies idempotent 1-stable range. Since R is a local ring, idempotents can be lifted modulo $J(R)$. From Theorem 9, the result follows. $\square$

In general, every ring satisfying idempotent 1-stable range has stable range one, but the converse is not true as the following shows.

EXAMPLE 11. Let $R = \{m/n \in \mathbb{Q} \mid 2 \nmid n \text{ and } 3 \nmid m(m/n \text{ in lowest terms})\}$. Then R is a semilocal ring, while idempotents do not lift modulo $J(R)$. So R has stable range one, but R does not satisfy idempotent 1-stable range from Theorem 9.

Let R be an associative ring with identity 1. Right R -module A is said to have finite exchange property if for every right R -module K and any two decompositions,

$$K = M \oplus N = \bigoplus_{i \in I} A_i, \quad (51)$$

where $M_R \cong A$ and the index set I is finite, there exist submodules $A'_i \subseteq A_i$ such that

$$K = M \oplus \left(\bigoplus_{i \in I} A'_i \right). \quad (52)$$

We call a ring R is a (right) weakly P -exchange ring if every right R -module has finite exchange property (cf. [12]). It is well known that regular rings, right perfect rings and weakly right perfect rings are all weakly P -exchange, while there still exist weakly P -exchange rings which belong to none of the above classes ([12, Ex. 4.6]). R is called to be exchange if right R -module R has finite exchange property. We know that regular rings, π -regular rings, unital C^* -algebras of real rank zero, semiperfect rings, left or right continuous rings and clean rings are all exchange. In [16], H. P. Yu proved that every exchange ring with all idempotents central has stable range one. Now we generalize this result as follows.

THEOREM 12. *Let R be a ring with all idempotents central. Then the following are equivalent:*

- (1) R satisfies idempotent 1-stable range.
- (2) R is a clean ring.
- (3) R is an exchange ring.

PROOF. (1) $\Rightarrow$ (2) Given any $a \in R$. From $aR + (-1)R = R$, we have an idempotent $e \in R$ such that $a + (-1)e = u$, and then $a = e + u$. So R is clean.

(2) $\Rightarrow$ (3) is clear from [11, Prop. 1.8, Thm. 2.1].

(3) $\Rightarrow$ (1) Assume that R does not satisfies idempotent 1-stable range. By Proposition 2, there exist $a, b \in R$ with $aR + bR = R$, while $a + bp \notin U(R)$ for any $p = p^2 \in R$.

Let $\Omega = \{A \mid A \text{ is a two-sided ideal of } R \text{ such that } a + bq \text{ is not a unit modulo } A \text{ for any } q = q^2 \in R\}$. It is easy to check that Ω is a nonempty inductive set. By using Zorn's

lemma, we have a two-sided ideal Q of R such that it is maximal in Ω .

By the maximality of Q , we show that R/Q is indecomposable as a ring. Given any $x \in R/Q$. Since R is exchange, so is R/Q . By [11, Thm. 1.1], there an idempotent $e \in R/Q$ such that

$$e \in x \left(\frac{R}{Q} \right), \quad 1 - e \in (1 - x) \left(\frac{R}{Q} \right), \quad (53)$$

and an idempotent $f \in R/Q$ such that

$$f \in \left(\frac{R}{Q} \right) x, \quad 1 - f \in \left(\frac{R}{Q} \right) (1 - x). \quad (54)$$

Since idempotents in R/Q can be lifted modulo Q , we may assume that e and f are both central idempotents in R/Q . So $e = 0$ or $e = 1$ and $f = 0$ or $f = 1$. Thus we see that x or $1 - x$ is right invertible in R/Q . Similarly, x or $1 - x$ is left invertible.

Assume that $x \in R/Q$ is not invertible. If x is not left invertible in R/Q , then rx is not left invertible for any $r \in R/Q$. Thus $1 - rx$ is left invertible, whence rx is left quasi-regular. This shows that $x \in J(R/Q)$. If x is not right invertible in R/Q , similarly to the discussion above, we have $x \in J(R/Q)$. So $J(R/Q) = \{x \in R/Q \mid x \text{ is not invertible in } R/Q\}$. This implies that R/Q is local. By virtue of Example 10, we claim that R/Q satisfies idempotent 1-stable range, a contradiction. Hence the result follows. $\square$

Theorem 12 shows that exchange rings with all idempotents central satisfy idempotent 1-stable range. Now we give an exchange ring R with noncentral idempotents, while it indeed satisfy idempotent 1-stable range.

EXAMPLE 13. Let

$$R = \begin{pmatrix} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}. \quad (55)$$

By [17, Ex. 3.10], R is an exchange ring with noncentral idempotents. According to Theorem 5, we directly verify that R satisfies idempotent 1-stable range.

Recall that a ring R is said to be strongly π -regular if every descending chain of right ideals of the form

$$aR \supseteq a^2R \supseteq a^3R \supseteq \cdots, \quad a \in R \quad (56)$$

becomes stationary. It is well known that every strongly π -regular ring is clean. Now we generalize this fact as follows.

COROLLARY 14. *Let R be a strongly π -regular ring. If $x, y \in R$ with $xy = yx$, then there exists an idempotent $e \in R$ such that*

$$xy + xe + 1 \in U(R). \quad (57)$$

PROOF. Given any $x, y \in R$ with $xy = yx$. Let S be an additive subgroup generated by the set

$$\{x^m y^n \mid m, n \geq 0\}. \quad (58)$$

Then S is a commutative subring of R . By virtue of [3, Cor. 1.10], we can find a commutative strongly π -regular subring T of R which contains S .

By Theorem 12, T satisfies idempotent 1-stable range with $x, y \in T$. Thus we can find

$$f = f^2 \in T \subseteq R \quad (59)$$

such that

$$x(y+1) - xf + 1 \in U(T) \subseteq U(R). \quad (60)$$

Let $e = 1 - f$. Then we have idempotent $e = e^2 \in R$ such that $xy + xe + 1 \in U(R)$, as desired. $\square$

A ring R is said to be right (left) quasi-duo if every maximal right (left) ideal is two-sided. By an argument of H. P. Yu, every weakly P -exchange ring with all idempotents central is right (left) quasi-duo. In general, the converse is not true such as

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}, \quad \text{where } F \text{ is a field.} \quad (61)$$

Now we give a theorem which guarantees the existence of one large class of rings satisfying idempotent 1-stable range.

THEOREM 15. *Let R be a right or left quasi-duo weakly P -exchange ring. Then R satisfies idempotent 1-stable range.*

PROOF. By [15, Prop. 2.1(1)], right primitive right quasi-duo rings are division. So every right or left quasi-duo ring has primitive factors artinian. Let Q be a prime ideal of R . Since R is a weakly P -exchange ring, so is R/Q . Similarly to [12, Prop. 4.1(2)], the finite exchange property of $R^{(\mathbb{N})}$ forces $J(R/Q)$ to be T -nilpotent. Assume that

$$0 \neq a \in J\left(\frac{R}{Q}\right). \quad (62)$$

Then there exist $x_1, x_2, \dots, x_n, \dots \in R/Q$ such that

$$ax_1a \neq 0, \quad ax_2ax_1a \neq 0, \dots, \quad ax_n \cdots ax_1a \neq 0, \dots, \quad (63)$$

a contradiction. Thus $J(R/Q) = 0$. So R/Q is an indecomposable exchange ring with primitive factors artinian and $J(R/Q) = 0$. Using [17, Lem. 3.7], we claim that R/Q is simple artinian. Thus R is an exchange ring with prime factors artinian, so it is strongly π -regular. Using [17, Thm. 3.8], we see that $R/J(R)$ is a regular ring with all idempotents central. From Theorem 12, $R/J(R)$ satisfies idempotent 1-stable range. As idempotents can be lifted modulo $J(R)$, we obtain the result from Theorem 9. $\square$

Recall that $p(a) = a$, $p(a, b) = 1 + ab$ and $p(a, b, c) = a + c + abc$ for any $a, b, c \in R$. $W(R)$ denotes the subgroup of $U(R)$ generated by

$$\{p(a, b, c)p(c, b, a)^{-1} \mid p(a, b, c) \in U(R), a, b, c \in R\}, \quad (64)$$

and $V(R)$ denotes the subgroup of $U(R)$ generated by

$$\{p(a, b)p(b, a)^{-1} \mid p(a, b) \in U(R), a, b \in R\}. \quad (65)$$

It is easy to verify that

$$p(a, b, c) = p(a, b)c + p(a), p(a, b, c)p(b, a) = p(a, b)p(c, b, a) \quad (66)$$

and

$$\begin{pmatrix} * & * \\ p(a, b, c) & * \end{pmatrix} = B_{21}(a)B_{12}(b)B_{21}(c). \quad (67)$$

We end this note by investigating Whitehead groups of rings with many idempotents.

THEOREM 16. *Let R satisfy idempotent 1-stable range. Then*

$$K_1(R) \cong \frac{U(R)}{V(R)}. \quad (68)$$

PROOF. For any $a, b, c \in R$ with $p(a, b, c) \in U(R)$, we see that $p(c, b, a) \in U(R)$. By virtue of Theorem 5, there exists an idempotent $e \in R$ such that $1 + b(c - e) \in U(R)$. Let $c - e = t$. Then $c = t + e$ and $1 + bt \in U(R)$. Observing that

$$\begin{aligned} \begin{pmatrix} * & * \\ p(a, b, c) & * \end{pmatrix} &= B_{21}(a)B_{12}(b)B_{21}(c) \\ &= (B_{21}(a)B_{12}(b)B_{21}(t))B_{21}(e) \\ &= B_{21}(a)[1 + bt, 1]B_{21}(t)B_{12}(b)[1, (1 + tb)^{-1}]B_{21}(e) \\ &= [1 + bt, 1]B_{21}(a + t + abt)B_{12}(b)[1, (1 + tb)^{-1}]B_{21}(e) \\ &= [1 + bt, (1 + tb)^{-1}]B_{21}((1 + tb)(a + t + abt))B_{12}(b(1 + tb)^{-1})B_{21}(e) \\ &= \begin{pmatrix} * & * \\ (1 + tb)^{-1}p((1 + tb)(a + t + abt), b(1 + tb)^{-1}, e) & * \end{pmatrix}. \end{aligned} \quad (69)$$

Thus we have

$$p(a, b, c) = (1 + tb)^{-1}p((1 + tb)(a + t + abt), b(1 + tb)^{-1}, e). \quad (70)$$

Analogously to [10, Thm. 1.6], we know that

$$\begin{aligned} p(a, b, c) &\equiv (1 + tb)^{-1}p(e, b(1 + tb)^{-1}, (1 + tb)(a + t + abt)) \pmod{V(R)} \\ &= (1 + tb)^{-1}(p(e, b(1 + tb)^{-1})(1 + tb)(a + t + abt) + p(e)) \\ &= (1 + tb)^{-1}(p(e, b(1 + tb)^{-1})p(t, b)p(a, b, t) + p(e)) \\ &= (1 + tb)^{-1}(p(e, b(1 + tb)^{-1})p(t, b, a)p(b, t) + p(e)). \end{aligned} \quad (71)$$

Similarly, we can verify that

$$\begin{aligned} p(c, b, a) &= p(e, (1 + bt)^{-1}b, (t + a + tba)(1 + bt))(1 + bt)^{-1} \\ &= (p(e, (1 + bt)^{-1}b)(t + a + tba)(1 + bt) + p(e))(1 + bt)^{-1} \\ &= (p(e, (1 + bt)^{-1}b)p(t, b, a)p(b, t) + p(e))(1 + bt)^{-1}. \end{aligned} \quad (72)$$

It is easy to check that

$$b(1+tb)^{-1} = (1+bt)^{-1}b. \quad (73)$$

Consequently, we have

$$(1+tb)p(a,b,c) \equiv p(c,b,a)(1+bt) \pmod{V(R)}. \quad (74)$$

Thus

$$p(a,b,c)(p(c,b,a))^{-1} \in V(R). \quad (75)$$

Therefore we conclude that

$$K_1(R) \cong \frac{U(R)}{W(R)} \cong \frac{U(R)}{V(R)}, \quad (76)$$

as asserted. $\square$

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