WEBBED SPACES, DOUBLE SEQUENCES, AND THE MACKEY CONVERGENCE CONDITION

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ABSTRACT. In [3], Gilsdorf proved, for locally convex spaces, that every sequentially webbed space satisfies the Mackey convergence condition. In the more general frame of topological vector spaces, this theorem and its inverse are studied. The techniques used are double sequences and the localization theorem for webbed spaces.

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1. Introduction. A web \( W \) in a topological vector space \( E \) is a countable family of balanced subsets of \( E \), arranged in layers. The first layer of the web consists of a sequence \( (A_p : p = 1, 2, \ldots) \) whose union absorbs each point of \( E \). For each set \( A_p \) of the first layer, there is a sequence \( (A_{pq} : q = 1, 2, \ldots) \) of sets, called the sequence determined by \( A_p \), such that

\[
A_{pq} + A_{pq} \subset A_p \quad \text{for each } q;
\]

\[
\bigcup \{ A_{pq} : q = 1, 2, \ldots \} \text{ absorbs each point of } A_p.
\]

Further, layers are made up in a corresponding way such that each set of the \( k \)th layer is indexed by a finite row of \( k \) integers and, at each step, the above mentioned two conditions are satisfied. Suppose that one chooses a set \( A_p \) from the first layer, then a set \( A_{pq} \) of the sequence determined by \( A_p \) and so on. The resulting sequence \( S = (A_p, A_{pq}, A_{pqr}, \ldots) \) is called a strand. Whenever we are dealing with only one strand, we can simplify the notation by writing \( W_1 = A_p, W_2 = A_{pq}, \text{ etc.} \) Thus, \( S = (W_k) \) is a strand, where, for each \( k, W_k \) is a set of the \( k \)th layer.

Let \( S = (W_k) \) be a strand. Consider \( x_k \in W_k \) and the series \( \sum_{k=1}^{\infty} x_k \). The space \( E \) is webbed if the series \( \sum_{k=1}^{\infty} x_k \) is convergent for any choice of \( x_k \in W_k \); and \( E \) is strictly webbed if \( \sum_{k=n+1}^{\infty} x_k \) converges to some \( x \in W_n \) for every \( n \in \mathbb{N} \) and for any choice of \( x_k \in W_k \). The standard references for webs in a topological vector space are [5, 7, 8].

Let \( (E, \tau) \) be a topological vector space. \( (x_n)_n \in E \) is a Mackey null sequence if there exists a sequence of real numbers \( (r_n)_n \) such that \( r_n \to \infty \) and \( r_n x_n \to 0 \) in \( E \). We say that \( (x_n)_n \in E \) is Mackey convergent to \( x \) if \( (x_n - x)_n \) is a Mackey null sequence. A topological vector space \( E \) satisfies the Mackey convergence condition (M.c.c.) if every null sequence is Mackey null.
2. Double sequences. A completing double sequence in a topological vector space \((E, \tau)\) is a family \((K^n_j)_{n,j \in \mathbb{N}}\) of balanced subsets such that

1. \(K^n_j \subset K^n_{j+1}\) for every \(n, j\) natural numbers;
2. \(K^n_{j+1} + K^n_{j+1} \subset K^n_j\) for every \(n, j\) natural numbers;
3. \(\bigcup_{n \in \mathbb{N}} K^n_j\) is absorbent in \(E\) for every \(j\) natural number;
4. for every \(j_0 \in \mathbb{N}\), if \(x_j \in K^n_j\) with \(j \geq j_0\), then \(\sum_{j=j_0+1}^\infty x_j\) converges in \(E\) to some \(x \in K^n_j\).

Moreover, \((K^n_j)_{j,n \in \mathbb{N}}\) is compatible with the topology if, for each zero neighborhood \(U\) in \(E\) and for every natural number \(n\), there exists a natural number \(J\) such that \(K^n_j \subset U\) for every \(j \geq J\).

For example, if \(E\) is sequentially complete and has a fundamental sequence of closed bounded sets \(A_1 \subset A_2 \subset \cdots\) such that, for each bounded set \(B \subset E\), there exists \(n_0 \in \mathbb{N}\) such that \(B \subset A_{n_0}\) (this is the case if \(E\) is the strong dual of a metrizable space). In this case, we define \(K^n_j = 2^{-j}A_n\) and it is easy to verify the properties (1) to (4), above. The reader can find further information concerning double sequences in [6].

A topological vector space \((E, \tau)\), with a compatible completing double sequence \((K^n_j)\), has a Sequential Double Sequence or the SDS property if, for each \(x_m \to 0\) in \(E\), there exists \(n_0 \in \mathbb{N}\) such that, for each \(j\), there exists a natural number \(M_j\) such that \(x_m \in K^{n_0}_j\), for every \(m \geq M_j\).

**Theorem 1.** Let \((E, \tau)\) be a topological vector space with the SDS property. Then \(E\) satisfies the Mackey convergence condition.

**Proof.** Let \(x_m \to 0\) in \((E, \tau)\). Let \((K^n_j)\) be a sequential double sequence, then there exists \(n_0 \in \mathbb{N}\) such that, for every \(j\), there exists a natural number \(M_j\) such that \(x_m \in K^{n_0}_j\), for every \(m \geq M_j\). For \(n, j \in \mathbb{N}\), we have \(K^n_{j+1} \subset (1/2)K^n_j\); so \(K^n_{j+2} \subset (1/2^2)K^n_j\). Consequently, for each \(l \in \mathbb{N}\), \(K^n_{j+l} \subset (1/2^l)K^n_j\). Note that \((1/2^l) \leq (1/j)\), for every \(j \in \mathbb{N}\) and \(K^{n_0}_{j+l} = K^{n_0}_{j+1} \subset (1/2)K^{n_0}_j \subset (1/j)K^{n_0}_j\). So, there exists \(M_{2j} \in \mathbb{N}\) such that \(x_m \in K^{n_0}_{2j} \subset (1/2)K^{n_0}_j \subset (1/j)K^{n_0}_j\), for every \(m \geq M_{2j}\) which implies that \(jx_m \in K^{n_0}_j\), for every \(m \geq M_{2j}\). Analogously, for \((j+1)\), there exists \(M_{2(j+1)} \geq M_{2j}\) such that \((j+1)x_m \in K^{n_0}_{j+1}\), for every \(m \geq M_{2(j+1)}\); and so, for all \(j \in \mathbb{N}\). Define \(r_m = j\) if \(M_{2j} \leq m < M_{2(j+1)}\), then \(\lim_{m \to \infty} r_m = \lim_{j \to \infty} j = \infty\). Since \((K^n_j)\) is compatible with the topology, we conclude that \(r_m x_m \to 0\).

From the theorem, a space with the SDS property is a space with the Mackey convergence condition. In what follows, we study the conditions under which we have an equivalence of these two properties. First, let us introduce another type of double sequences: a topological vector space \((E, \tau)\), with a compatible completing double sequence \((K^n_j)\), has a quasi-Sequential Double Sequence or the qSDS property if, for each \(x_n \to 0\) in \(E\), there exists \(n_0\) such that, for every \(j\), there exists a natural number \(M_j\) and a positive real number \(\alpha_j\) such that \(m > M_j\) implies that \(x_m \in \alpha_j K^{n_0}_j\).

If \(\alpha_j = 1\), for every \(j\), in a qSDS, then it becomes on SDS. So, the qSDS is more general than the SDS. The next proposition gives the condition for the equivalence.

**Proposition 2.** Let \((E, \tau)\) be a topological vector space with the Mackey convergence condition. Then the SDS and the qSDS are the same.
Proof. Let $x_m \to 0$ in a space $(E, \tau)$ with qSDS property. By the Mackey convergence condition, there exists a scalar sequence $r_m \to \infty$ such that $r_m x_m \to 0$. Then there exists $n_0$ such that $r_m x_m \in \alpha_j K_j^{n_0}$, for some $\alpha_j > 0$ whenever $m \geq M_j$. Hence, $x_m \in (\alpha_j/r_m)K_j^{n_0} \subset K_j^{n_0}$ if $m \geq M_j$ and $r_m \geq \alpha_j$.

Next, we see an example, where the qSDS property holds and the SDS property does not.

Let $(E, \| \cdot \|)$ be a Banach space with a sequence $(x_m)_{m \in \mathbb{N}}$ weakly convergent to zero and not norm convergent. Let $B$ be the closed unit ball in $E$. For each $n, j \in \mathbb{N}$, let $K_j^n = 2^{-j}B$. Then $(K_j^n)$ is a compatible completing double sequence with respect to the norm topology and, consequently, with respect to any weaker topology $\tau$, especially the weak topology since the map $i: (E, \| \cdot \|) \to (E, \tau)$ is continuous. Now, $(x_m)_{m \in \mathbb{N}}$ is not contained in $K_j^n$, since $K_j^n$ are neighborhoods in the norm topology such that $\bigcap_j K_j^n = \{0\}$ and, by [4, Ex. 4] and [4, cor. of Thm. 3], $(E, \sigma)$ does not have the M.c.c. Nevertheless, $(x_m)_{m \in \mathbb{N}}$ is bounded with respect to both the weak and norm topologies. So, for every $K_j^n$, there exists $\alpha_j$ such that $(x_m)_{m \in \mathbb{N}} \subset \alpha_j K_j^n$.

We have the following implication: SDS $\Rightarrow$ qSDS. This implication can be reversed if the space has the M.c.c. Furthermore SDS $\Rightarrow$ M.c.c. So, we have the following corollary:

**Corollary 3.** Let $E$ be a topological vector space with a compatible completing double sequence. Then $E$ has SDS property if and only if the qSDS property and M.c.c. hold.

3. Mackey convergence and sequentially webbed spaces. $E$ is sequentially webbed if it has a compatible web $W$ such that, for every null sequence $(x_n)_{n \in \mathbb{N}}$ in $E$, there exists a finite collection of strands $\{(W^{(1)}_k), \ldots, (W^{(m)}_k)\}$ of $W$ such that, for every natural number $k$, there exists $M_k$ such that $n \geq M_k$ implies $x_n \in \bigcup_{l=1}^{m} W_k^{(l)}$. Gilsdorf [3] proved two relations between the M.c.c. and the sequentially webbed spaces in the locally convex case.

Here, we generalize these results. One to topological vector spaces and the other to locally $r$-convex spaces. In fact, the concept of webbed spaces, introduced here, does not use local convexity. Note that in this case, in each strand, we have $2W_{k+1} \subset W_{k+1} + W_{k+1} \subset W_k$ so that $W_{k+1} \subset 2^{-1}W_k$, and then following the proof of [3, Thm. 12], we have: if $(E, \tau)$ is a sequentially webbed topological vector space, then $E$ has the M.c.c.

In order to obtain a converse of this result, we need to use a localization theorem [5, Thm. 5.6.3].

Let $0 < r \leq 1$ fixed. $A \subset E$ is $r$-convex if $\lambda A + \mu A \subset A$, for every $\lambda, \mu \geq 0$ such that $\lambda r + \mu r = 1$. Moreover, if $A$ is balanced, we say that $A$ is absolutely $r$-convex. If $r = 1$, we have the usual convexity definition.

For $U \subset E$ balanced and absorbent, let $q_u : E \to \mathbb{R}^+$ be the Minkowski functional defined by $x \to \inf \{ \rho > 0 : x \in \rho U \}$. $q_u$ is an $r$-seminorm if $q_u(x + y)^r \leq q_u(x)^r + q_u(y)^r$. Furthermore, if $q_u^{-1}(0) = 0$, it is called an $r$-norm. $(E, \tau)$ is locally $r$-convex if it has a fundamental system of zero neighborhoods formed by $r$-convex sets.

Now, we can use the $E_B$ spaces for locally $r$-convex spaces. $(E, \tau)$ locally $r$-convex space is locally $r$-Baire if, for every bounded set $A \subset E$, there exists $B$ absolutely $r$-convex and bounded such that $A \subset B$ and the space $(E_B, \rho_B)$ is a Baire space, where $E_B$
is the span of $B$ and $\rho_B$ is the topology generated by the $r$-norm $q_r^B$.

**Theorem 4.** Let $(E, \tau)$ be a locally $r$-Baire locally $r$-convex space and strictly webbed. If $E$ satisfies the Mackey convergence condition, then $E$ is sequentially webbed.

**Proof.** Let $W$ be a strict web in $E$; $(x_n)_n \subset E$ a null sequence, and $r_n \to \infty$ a sequence of real numbers such that $r_n x_n \to 0$ in $E$. Let $A = \{r_n x_n : n \in \mathbb{N}\}$, $A$ is bounded, then there exists a bounded absolutely $r$-convex set $B$ such that $(E_B, \rho_B)$ is a Baire space and $A$ is a bounded set in $E_B$. The identity map $i : E_B \to E$ is continuous. Hence, by the localization theorem, $i$ has a closed graph and there exists a strand $(W_k)$ such that $i^{-1}(W_k) = E_B \cap W_k$ is a zero neighborhood in $(E_B, \rho_B)$ for every $k$. Finally, $A \subset \alpha_k(E_B \cap W_k) \subset \alpha_kW_k$ for some $\alpha_k$, a positive real number. So, $r_n x_n \in \alpha_k W_k$ and $x_n \in (\alpha_k/r_n) W_k \subset W_k$, for $n$ sufficiently large such that $|\alpha_k/r_n| \leq 1$. \[ \square \]

**References**


