EXISTENCE OF GLOBAL SOLUTION FOR A DIFFERENTIAL SYSTEM WITH INITIAL DATA IN $L^p$

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Abstract. In this paper, we study the system governing flows in the magnetic field within the earth. The system is similar to the magnetohydrodynamic (MHD) equations. By establishing a new priori estimates and following Calderón’s procedure for the Navier Stokes equations [1], we obtained, for initial data in space $L^p$, the global in time existence and uniqueness of weak solution of the system subject to appropriate conditions.

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1. Introduction. We consider in this work the following differential system arising from geophysics (cf. Hide [7]), which governs the flow of an electrically-conducting fluid in the presence of a magnetic field, when referred to a frame which rotates with angular velocity $\Omega$ relative to an inertial frame

$$\begin{align*}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v &= \nu \Delta v + \frac{1}{\rho} \nabla p - \frac{2 \Omega}{\rho} \times (\nabla \times b) \times b + f(x), \\
\frac{\partial b}{\partial t} &= \lambda \Delta b - \nabla \times (v \times b) - \frac{1}{\mu} \nabla q + g(x),
\end{align*}$$

where $v$ is the Eulerian flow velocity, $\rho$ is the density, $b$ is the magnetic field, $p$ is the pressure, $\nu$, $\mu$ are, respectively, constants of kinematical viscosity, magnetic permeability, $\lambda = \eta/\mu$ with electrical resistivity $\eta$, and $f(x), g(x)$ are volume forces.

The initial conditions are as follows:

$$v(x, 0) = v_0, \quad b(x, 0) = b_0 \quad \text{for} \quad x \in \mathbb{R}^n. \quad (1.2)$$

The existence of solutions of system (1.1) and (1.2) in $L^2$ has been proved in [9]. Some regularity properties and large time behaviors of the solutions for a similar system, the MHD equations, are obtained in Sermange [10] and Temam [12]. More recently, we obtained in [2] the local in time existence and uniqueness of weak solutions of the system in $L^p$ with $p > n$.

Motivated by Calderón’s work on the Navier Stokes equations [1], we consider in this paper the initial value problem for the above system in the infinite cylinder $S = (0, \infty) \times \mathbb{R}^n$ with initial data $v_0, b_0 \in L^p$ with $p \leq n$.

This article is arranged in the following order: in Section 2, we introduce some notations and definitions. Applying Calderón’s partition lemma, we introduce in Section 3
Leray’s approximating system for our problem. In Section 4, we state and briefly prove some lemmas similar to those for Navier-Stokes equations. Finally, in Section 5, by establishing a priori estimates for our system and adapting Calderón’s technique, we prove the global in time existence and uniqueness of weak solution of (1.1) and (1.2) for initial data in $L^p$.

2. Notations and definition of weak solution. In this section, we introduce some notations and the definition of a weak solution of the differential system (1.1) and (1.2).

Denote by $\mathcal{L}^p,q(S_T)$ the standard functional space consisting of Lebesgue measurable vector functions $u = (u_1, u_2, \ldots, u_n)$ with the following property:

$$
\|u\|_{p,q} = \left(\int_0^T \left(\int_{\mathbb{R}^n} |u_j(x,t)|^p \, dx \right)^{q/p} \right)^{1/q} < \infty,
$$

(2.1)

where $S_T = (0, T) \times \mathbb{R}^n$. Let $u^* = \sup_t |u|$ and define $\|u^*\|_p(T) = (\int (\sup_{0 \leq t < T} |u|)^p \, dx)^{1/p}$.

Let $\mathcal{D}^p,q(S_T) = L^p,q(S_T) \times L^p,q(S_T)$ with the standard product norm $\|(v, b)\|_{p,q} = \|v\|_{p,q} + \|b\|_{p,q}$ and $L^p(R^n) = L^p(R^n) \times \cdots \times L^p(R^n)$ with the norm $\|g\|_p = \sum_{i=1}^n \|g_i\|_p$ for $g \in L^p(R^n)$.

Let $\mathcal{D}'(R^n)$ denote the space of rapidly decreasing functions on $R^n$, $\mathcal{D}'(R^n)$ the space of tempered distributions, and $\mathcal{D}_T$ the space of functions $\phi(x,t) = (\phi_1(x,t), \ldots, \phi_n(x,t))$ with the properties: $\phi_i \in \mathcal{D}'(R^{n+1})$, $\phi_i(x,t) = 0$ for $t \geq T$; $\text{div} \phi = \sum_{i=1}^n D_{x_i} \times \phi(x,t) = 0$ for all $t$.

**Definition 2.1.** A function $u = (v, b)$ is a weak solution of (1.1) and (1.2) with initial divergence free data $(v_0, b_0) \in \mathcal{D}'(R^n)$ if the following conditions hold

1. $u(x,t) \in \mathcal{D}^p,q(S_T)$ for some $p, q$ with $p, q \geq 2$;
2. for $\phi, \psi \in \mathcal{D}_T$,

$$
\int_0^T \int_{\mathbb{R}^n} \langle v, (v \Delta + D_t) \phi \rangle \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} \langle v, (\nabla \phi) v \rangle \, dx \, dt
$$

$$+ \int_0^T \int_{\mathbb{R}^n} \langle v, 2\Omega, \times \phi \rangle \, dx \, dt - \frac{1}{\rho \mu} \int_0^T \int_{\mathbb{R}^n} \langle b, (\nabla \phi) b \rangle \, dx \, dt
$$

$$= -\int_{\mathbb{R}^n} \langle v_0, \phi(x,0) \rangle \, dx + \int_0^T \int_{\mathbb{R}^n} \langle f(x,t), \phi \rangle \, dx \, dt;
$$

$$
\int_0^T \int_{\mathbb{R}^n} \langle b, (\lambda \Delta + D_t) \psi \rangle \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} \langle v, (\nabla \psi) b \rangle \, dx \, dt
$$

$$- \int_0^T \int_{\mathbb{R}^n} \langle b, (\nabla \psi) v \rangle \, dx \, dt
$$

$$= -\int_{\mathbb{R}^n} \langle b_0, \psi(x,0) \rangle \, dx + \int_0^T \int_{\mathbb{R}^n} \langle g(x,t), \psi \rangle \, dx \, dt;
$$

(2.2)

3. for almost every $t \in [0, T]$, $\text{div} v(x,t) = \text{div} b(x,t) = 0$ in the distributional sense.

Following Fabes et al. [4], we can find a divergence free matrix fundamental solution $E_{i,j}$ for the $n$-dimensional heat equation. We define matrices $(E_{ik}^i)$, $k = 1, 2$ as follows:
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\[ E_{i,j}^k = \delta_{i,j} \Gamma_k(x,t) - R_i R_j \Gamma_k(x,t), \] (2.3)

where

\[ \Gamma_1 = \frac{e^{-|x|^2/4vt}}{(4\pi vt)^{n/2}}, \quad \Gamma_2 = \frac{e^{-|x|^2/4\lambda t}}{(4\pi \lambda t)^{n/2}}, \] (2.4)

\( R_j \) is the jth Riesz transform, namely, \( R_j \) is a singular integral operator on \( L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), defined as

\[ R_j(f) = \text{P.V.} C_j \int_{\mathbb{R}^n} (x_j - y_j) |x - y|^{-n-1} f(y) dy. \] (2.5)

Now, we define an integral operator \( A(v,w) \) for \( v = (v_1, \ldots, v_n) \), \( w = (w_1, \ldots, w_n) \).

Denote

\[ B_k(v,w)(x,t) = \int_0^t \int_{\mathbb{R}^n} \langle v(y,s), \nabla E^k(x - y, t - s) \rangle w(y,s) dy ds; \quad \text{for } k = 1, 2. \] (2.6)

\[ D(v)(x,t) = \int_0^t \int_{\mathbb{R}^n} \langle v(y,s), 2\Omega \times E^1(x - y, t - s) \rangle dy ds. \] (2.7)

For \( u_1 = (v_1, b_1) \), \( u_2 = (v_2, b_2) \), let

\[ A(u_1, u_2) = \begin{pmatrix} B_1(v_1, v_2) - \frac{1}{\rho \mu} B_1(b_1, b_2) \\ \frac{1}{2} \left[ B_2(v_1, b_1) - B_2(b_1, v_1) + B_2(v_2, b_2) - B_2(b_2, v_2) \right] \end{pmatrix}. \] (2.8)

3. Approximating system. The following result was obtained in [2].

**Theorem 3.1.** Let \( v_0, \ b_0 \in L^r \), \( 1 \leq r < \infty \), be divergence free weakly. \( u(x,t) = (v(x,t), b(x,t)) \in L^{p,q}(S_T), \ p, q \geq 2, \ p < \infty \), is a weak solution of (1.1) and (1.2) with initial value \( (v_0, b_0) \) if and only if \( u \) is a solution of the following integral equation:

\[ u + A(u,u) + D(u) = u^0 + f^0, \] (3.1)

where

\[ u^0 = \begin{pmatrix} \int_{\mathbb{R}^n} \Gamma_1(x - y, t) v_0(y) dy \\ \int_{\mathbb{R}^n} \Gamma_2(x - y, t) b_0(y) dy \end{pmatrix}, \]

\[ f^0 = \begin{pmatrix} \int_0^t \int_{\mathbb{R}^n} E^1(x - y, t - s) f(y,s) dy ds \\ \int_0^t \int_{\mathbb{R}^n} E^2(x - y, t - s) g(y,s) dy ds \end{pmatrix}. \] (3.2)

We need the following lemmas obtained by Calderón [1].

**Lemma 3.1.** Let \( f \in L^p(\mathbb{R}^n), \ 2 < p < n \), be a given vector function such that \( \text{div} \ f = 0 \) in the distributional sense. Then, for each \( s > 0 \), \( f \) can be expressed as \( g + h \), where

\[ ||g||_n \leq c s^{1-(p/n)} ||f||_p^{p/n}, \quad \text{div} \ g = 0, \]

\[ ||h||_2 \leq c s^{1-(p/2)} ||f||_p^{p/2}, \quad \text{div} \ h = 0, \] (3.3)

where the constant \( c \) depends only on \( n \) and \( p \).
Lemma 3.2. Let \( T(u, v) = B(u, v) + l(u) + F \) (\( B(u, v) \) is bilinear and \( l(u) \) is linear) satisfy

\[
\|T(u, v)\| \leq c_1 \|u\| \|v\| + c_2 \|u\| + \|F\| \quad (3.4)
\]

with the same norm in a Banach space. Then the quadratic operator \( T(u, v) \) maps the ball \( \{\|u\| \leq s_1\} \) into itself if \( s_1 \) is the smallest root of

\[
c_1 s^2 + (c_2 - 1)s + \|F\| = 0,
\]

provided that \( c_1, c_2, \) and \( \|F\| \) satisfy

\[
(1 - c_2)^2 > 4c_1 \|F\|, \quad c_1 > 0, \quad 0 \leq c_2 < 1.
\]

If \( 2s_1c_1 + c_2 < 1, \) \( T(u, v) \) is a contraction mapping in the ball of radius \( s_1. \) In particular, \( T(u, v) \) is a contraction mapping in the ball of radius \( s_1 \) if

\[
2c_1 \|F\| ((1 - c_2)^2 - 4c_1 \|F\|)^{-1/2} + c_2 < 1.
\]

Consider the following system in \( v_1, v_2, b_1, b_2, p_1, p_2, q_1, \) and \( q_2 \)

\[
L_1 v_1 + (\nabla v_1) v_1 - \frac{1}{\rho \mu} (\nabla b_1) b_1 + \nabla p_1 = 0,
\]

\[
L_1 v_2 + (\nabla v_2) v_2 + (\nabla v_2) v_1 + (\nabla v_1) v_2 - \frac{1}{\rho \mu} (\nabla b_2) b_2 + (\nabla b_2) b_1 + (\nabla b_1) b_2 + \nabla p_2 = 0,
\]

\[
L_2 b_1 + (\nabla b_1) v_1 - (\nabla v_1) b_1 + \nabla q_1 = 0,
\]

\[
L_2 b_2 + (\nabla b_2) v_2 + (\nabla b_2) v_1 + (\nabla b_1) v_2 - (\nabla v_2) b_2 - (\nabla v_1) b_1 - (\nabla v_1) b_2 + \nabla q_2 = 0,
\]

\[
\text{div} v_i = 0, \quad \text{div} b_i = 0, \quad i = 1, 2,
\]

\[
v_i(x, 0) = h_i(x), \quad b_i(x, 0) = k_i(x), \quad i = 1, 2
\]

(3.8)

where \( L_1 = \partial / \partial t - \nu \Delta, L_2 = \partial / \partial t - \lambda \Delta. \) We have the following definition.

Definition 3.1. The vector \( ((v_1, v_2), (u_1, u_2)) \) is said to be a weak solution of (3.8) if \( ((v_1 + v_2), (b_1 + b_2)) \) is a weak solution of (1.1) and (1.2) with initial data \( (h_1 + h_2, k_1 + k_2). \)

It then follows from Theorem 3.1 that

Theorem 3.2. The vector functions \( ((v_1, v_2), (b_1, b_2)) \) \( \in \mathcal{L}^{p,q}(S_T)^4, \) \( 2 \leq p, q \leq \infty, \) are weak solutions of (3.8) if and only if they are solutions of the following integral equations:

\[
v_1 + B_1(v_1, v_1) - \frac{1}{\rho \mu} B_1(b_1, b_1) = v_1^0,
\]

\[
v_2 + B_1(v_2, v_2) + B_1(v_1, v_2) + B_1(v_2, v_1) - \frac{1}{\rho \mu} [B_1(b_2, b_2) + B_1(b_1, b_2) + B_1(b_2, b_1)] = v_2^0,
\]

\[
b_1 + B_2(v_1, b_1) - B_2(b_1, v_1) = b_1^0,
\]

\[
b_2 + B_2(v_2, b_2) + B_2(v_1, b_2) + B_2(v_2, b_1) - [B_2(b_2, v_2) + B_2(b_1, v_2) + B_2(b_2, v_1)] = b_2^0,
\]

(3.9)
where

\[ v_i^0 = \Gamma_1 * h_i, \quad b_i^0 = \Gamma_2 * k_i, \quad i = 1, 2. \] (3.10)

Now, let us introduce Leray’s approximating system. Let \( \alpha(x) \) be a \( C^m \) nonnegative, compact supported function on \( \mathbb{R}^n \) with integral equal to 1, \( \alpha_\varepsilon(x) = \varepsilon^{-m} \alpha(\varepsilon^{-1} x) \).

Denote the modifying function of \( u(x,t) \) by \( u^\varepsilon(x,t) \), i.e., \( u^\varepsilon = \alpha_\varepsilon * u \). For each \( \varepsilon \), consider the following approximating system

\[
L_1 v_1 + (\nabla u) v_1^\varepsilon - \frac{1}{\rho \mu} (\nabla b_1) b_1^\varepsilon + \nabla p_1 = 0 \tag{3.11}
\]

\[
L_1 v_2 + \nabla (v_1 + v_2) v_2^\varepsilon + (\nabla v_2) v_2^\varepsilon - \frac{1}{\rho \mu} ((\nabla b_2) b_2^\varepsilon \\
+ (\nabla b_2) b_2^\varepsilon) + \nabla p_2 = 0, \tag{3.12}
\]

\[
L_2 b_1 + (\nabla b_1) v_1^\varepsilon - (\nabla v_1) b_1^\varepsilon + \nabla q_1 = 0, \tag{3.13}
\]

\[
L_2 b_2 + (\nabla b_2) v_2^\varepsilon + (\nabla b_2) v_2^\varepsilon - (\nabla v_2) b_2^\varepsilon \\
- (\nabla v_2) b_2^\varepsilon + b_2^\varepsilon + \nabla q_2 = 0, \tag{3.14}
\]

\[
\text{div} v_i = 0, \quad \text{div} b_i = 0, \quad i = 1, 2, \tag{3.15}
\]

\[
v_i(x,0) = v_i^\varepsilon(x), \quad b_i(x,0) = b_i^\varepsilon(x), \quad i = 1, 2, \tag{3.16}
\]

where \( v_i^\varepsilon, b_i^\varepsilon, i = 1, 2, \) are partitions of initial data \( v_0, b_0 \), respectively, in the sense of Lemma 3.1, i.e., \( v_0 = v_1^\varepsilon + v_2^\varepsilon, b_0 = b_1^\varepsilon + b_2^\varepsilon \). From Lemma 3.1, we have

\[
\begin{align*}
||v_1^\varepsilon||_n & \leq ||v_1^\varepsilon||_n \leq cs^{1-(p/n)} ||v_0||_n, \\
||v_2^\varepsilon||_2 & \leq ||v_2^\varepsilon||_2 \leq cs^{1-(p/2)} ||v_0||_2, \\
||b_1^\varepsilon||_n & \leq ||b_1^\varepsilon||_n \leq cs^{1-(p/n)} ||b_0||_n, \\
||b_2^\varepsilon||_2 & \leq ||b_2^\varepsilon||_2 \leq cs^{1-(p/2)} ||b_0||_2. \tag{3.17}
\end{align*}
\]

4. Some lemmas. In this section, we present some lemmas without providing much of the details of their proofs for the arguments involved are similar to those used in [1].

First, we consider (3.11), (3.13), and (3.15) with corresponding data \( v_1(x,0) = v_1^\varepsilon, b_1(x,0) = b_1^\varepsilon \). The problem is equivalent to the following integral equations (cf. [2])

\[
v_1 + B_1(v_1, v_1^\varepsilon) - \frac{1}{\rho \mu} B_1(b_1, b_1^\varepsilon) = v_1^0, \tag{4.1}
\]

\[
b_1 + B_2(v_1, b_1^\varepsilon) - B_2(b_1, v_1^\varepsilon) = b_1^0, \]

where \( v_1^0, v_2^0 \) are defined by (3.10) with \( h_i, k_i \) being replaced by \( v_1^\varepsilon, b_1^\varepsilon \), respectively. Denote \( u_1 = (v_1, b_1) \), the solution of (4.1). Define, for \( s > 0 \), and a function \( w, w^s = w \) if \( |w| < s, w = 0 \) otherwise. We have the following lemma:
**Lemma 4.1.** The system (4.1), including the limit case, i.e., when \( u_1 = u_1^r, (v_1^0, b_1^0) = (v_1^0, b_1^0) \), admits a unique solution \( u_1 = (v, b) \), for all \( t \), satisfying

\[
\|u_1^+\|_n(\infty) \leq cs^{1-(p/n)}\|u_1^0\|_{p/n},
\]

provided that \( s^{1-(p/n)}\|u_1^0\|_{p/n} < \varepsilon_0 \), where \( u_1^0 = (v_1^0, b_1^0) \), and

\[
\|u_1^+\|_p(\infty) \leq c \max \left( s^{1-(p/n)}\|u_1^0\|_{p/n}, \|u_1^0\|_p \right),
\]

provided that \( \max(s^{1-(p/n)}\|u_1^0\|_{p/n}, \|u_1^0\|_p) < \varepsilon_0 \), where \( (u_1^0)^\varepsilon = ((v_1^0)^\varepsilon, (b_1^0)^\varepsilon) \), \( \varepsilon_0 \) is a fixed and a small constant and \( c \) depends only on \( \varepsilon_0 \).

**Proof.** The proof is a direct extension of that of [1, Lem. III.1].

**Lemma 4.2.** Let \( u_1' = (v_1', b_1') \) be chosen such that \( \|u_1'(t)\|_p \) is so small that the existence of solution \( u_1 \) is assured by Lemma 4.1 and such that, for all \( t \),

\[
\|u_1^+\|_n < a_0 < c_0^{-1},
\]

where \( c_0 \) is an independent constant. Suppose that \( u_2 = (v_2, b_2) \) is a solution of (3.12), (3.14), (3.15), and (3.16) and suppose that \( \nabla v_2, \nabla b_2, (\partial/\partial t)v_2, (\partial/\partial t)b_2 \in L^2(S_T) \). Then \( u_2 \) satisfies the following estimate:

\[
\|u_2(t)\|_2^2 + 2(1 - c_0a_0)\int_0^t \|\nabla u_2\|_2^2 dt \leq \|u_2(0)\|_2^2,
\]

where

\[
\|u_2\|_2^2 = \left( \|v_2\|_2^2 + \frac{1}{\rho \mu}\|b_2\|_2^2 \right),
\]

\[
\|\nabla u_2\|_2^2 = \left( \|\nabla v_2\|_2^2 + \frac{1}{\rho \mu}\|\nabla b_2\|_2^2 \right).
\]

**Proof.** Multiplying (3.12) and (3.14) by \( v_2, b_2 \), respectively and integrating over \( R^n \), we get

\[
\frac{1}{2} \frac{d\|v_2\|_2^2}{dt} + \nu \|\nabla v_2\|_2^2 + ((\nabla v_1)v_2^\varepsilon, v_2) + \frac{1}{\rho \mu} \left((\nabla b_1)b_2^\varepsilon + ((\nabla b_2)b_1^\varepsilon, v_2) + ((\nabla b_2)b_1^\varepsilon, v_2) \right) = 0,
\]

\[
\frac{1}{2} \frac{d\|b_2\|_2^2}{dt} + \lambda \|b_2\|_2^2 + ((\nabla b_1) b_2^\varepsilon, b_2) + \left[((\nabla v_1)b_2^\varepsilon, b_2) + ((\nabla v_2)b_1^\varepsilon, b_2) - ((\nabla v_2)b_1^\varepsilon, b_2) \right] = 0.
\]

Note that, for functions \( a, b, c, \) and exponents \( r, n, 2 \) such that \( (1/r) + (1/n) + (1/2) = 1 \), we have

\[
\|((\nabla a)b, c)\| \leq \|\nabla a\|_2\|b\|_r\|c\|_n.
\]

Multiplying (4.8) by \( (1/\rho \mu) \) and adding the resulting equation to (4.7), we obtain
Taking $\epsilon$ satisfying (4.1). Then, for all $i(u_1, u_2)$, we can prove that
\begin{align}
\frac{1}{2} \frac{d}{dt} \| (v_2, b_2) \|^2 + \| (\nabla v_2, \nabla b_2) \|^2_2 \\
\leq c_1 (\| v_1 \|_n + \| b_1 \|_n) (\| \nabla v_2 \|^2_2 + \| \nabla b_2 \|^2_2) \\
\leq c_2 (\| v_1 \|_n + \| b_1 \|_n) \| (v, b) \|_2^2.
\end{align}
(4.10)

It is then standard to obtain (4.5).
\hfill \Box

Now, let us consider the existence of a weak solution of (3.12), (3.14), (3.15), and (3.16). It is easy to see that the system is equivalent to the following
\begin{align}
v_2 + B_1 (u_1, u_2) &= v_2^0, \\
b_2 + B_2 (u_1, u_2) &= b_2^0,
\end{align}
(4.11)

where
\begin{align}
B_1 (u_1, u_2) &= B_1 (v_2, v_2^0) + B_1 (v_1, v_2^0) + B_1 (v_2, v_1^0) \\
&\quad - \frac{1}{\rho \mu} B_1 (b_2, b_2^0) + B_1 (b_1, b_2^0) + B_1 (b_2, b_1^0), \\
B_2 (u_1, u_2) &= B_2 (v_2, b_2^0) + B_2 (v_1, b_2^0) + B_2 (v_2, b_1^0) \\
&\quad - B_2 (b_2, v_2^0) + B_2 (b_1, v_2^0) + B_2 (b_2, v_1^0).
\end{align}
(4.12)

**Lemma 4.3.** If $T$ is suitably small, then there exists a solution $u_2$ of (4.11) such that
\begin{align}
\| u_2^* \|_2 (T) < \infty.
\end{align}
(4.13)

**Proof.** Applying the standard estimate on $E^i$, the definition of $B_i$ (cf. (2.6)) and the Hardy-Littlewood-Sobolov potential inequality, we can prove that
\begin{align}
\| B_i (u_1, u_2) \|_2 (T) \leq c \left( \epsilon^{-n/2} \| u_2^* \|_2 (T) + \| u_1^* \|_n (T) \right) \| u_2^* \|_2 (T), \quad i = 1, 2.
\end{align}
(4.14)

Taking $\epsilon^{-n/2} T^{1/2}$ and $\| u_1^* \|_n$ small enough, we can apply Lemma 3.2 to obtain the existence of $u_2$.
\hfill \Box

Using the arguments in the proofs of Lemmas [1, III.3, III.4], one can similarly prove the next two lemmas.

**Lemma 4.4.** Let $u_1 (x, t) = (v_1, b_1)$ be the solution obtained in Lemma 4.1 solving (4.1). Then, for all $T > 0$ and all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we have
\begin{align}
\| D^\alpha u_1 \|_n (S_T) < \infty, \\
\| D_t D^\alpha u_1 \|_n (S_T) < \infty.
\end{align}
(4.15)

**Lemma 4.5.** Consider the following integral equations of unknown $u_2 = (v_2, b_2)$:
\begin{align}
v_2 + B_1 (u_1, u_2) &= F_1 (x, t), \\
b_2 + B_2 (u_1, u_2) &= F_2 (x, t),
\end{align}
(4.16)

where $B_i$, $i = 1, 2$ are defined in (4.12), $u_1$ is the solution of (4.1), and $F_1, F_2$ are functions satisfying
\begin{align}
\| D^\alpha F_i \|_2 (S_T) < \infty, \quad \| D_t D^\alpha F_i \|_2 (S_T) < \infty.
\end{align}
(4.17)
If we denote $T > 0$ the existence interval for $t$ of solution of (4.16) by the standard fixed point argument, then

$$\|D^\alpha u_2\|_2^2(S_T) < \infty,$$  \hspace{1cm} (4.18)

$$\|D_t D^\alpha u_2\|_2^2(S_T) < \infty.$$  \hspace{1cm} (4.19)

Using the above estimates, we can prove the following theorem.

**Lemma 4.6.** The solution obtained in Lemma 4.3 can be extended to all time $t > 0$ and it satisfies (4.5) for all $t$.

**Proof.** We only give a sketch of the proof here. The existence time $T$ obtained in Lemma 4.3 by the standard fixed point argument depends only on the $L^2$ norm of the initial data and $\|u_1\|_n$. Lemma 4.2 implies that $\|u_2(t)\|_2$ is uniformly bounded by the corresponding norm of the initial data when $u_2$ satisfies the regularity conditions of Lemma 4.2, which is guaranteed by Lemmas 4.4 and 4.5. Therefore, (4.5) holds for all $t$ by moving from $[0, T]$ to $[T, 2T]$ to $[2T, 3T]$ and so on. And then the interval of existence can be extended to $(0, \infty)$.

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5. The global existence theorem. In this section, we establish some a priori estimate for the solution of (4.11) and obtain, by following Calderón's procedure [1], the global existence and uniqueness of solution of (1.1) and (1.2).

To adapt Leray’s argument [8] to prove Lemmas 5.2 and 5.3 that we state later, we need to establish the following a priori estimate.

**Lemma 5.1.** For a $C^\infty$ function $\beta(x)$ satisfying $\beta(x) = 1$, if $|x| > N$; $\beta(x) = 0$, if $|x| < N/2$, and $\|\nabla \beta\| \leq C/N$, the solution $u_2$ of (4.11) satisfies the following inequality

$$\frac{1}{2} \int_{R^n} \beta(x) \left[ |v_2|^2 + \frac{1}{\rho \mu} |b_2|^2 \right] dx + \int_0^t \int_{R^n} \beta(x) \left[ \nabla v_2^2 + \frac{1}{\rho \mu} |\nabla b_2|^2 \right] dx dt$$

$$\leq \frac{1}{2} \int_{R^n} \beta(x) \left[ |v_2(0)|^2 + \frac{1}{\rho \mu} |b_2(0)|^2 \right] dx + C \left( \left( \frac{1 + t}{N} \right) |u_2(0)|^2 \right) dx dt$$

$$+ \frac{C}{N} \|u_1\|_{n, \infty}(T) |u_2(0)|^2 + C \|\beta u_1\|_{n, \infty}(T) |u_2(0)|^2,$$  \hspace{1cm} (5.1)

where $|u_2|_2$ is defined by (4.6).

**Proof.** Multiplying equations (3.12) and (3.14) by $\beta v_2$ and $\beta b_2$, respectively, and integrating over $R^n$, we get

$$\frac{1}{2} \frac{d}{dt} \int_{R^n} \beta(x) |v_2|^2 dx + \nabla v_2 \cdot \nabla (\beta v_2) + \left( (\nabla v_1) v_2^\#, \beta v_2 \right)$$

$$+ \left( (\nabla v_2) v_2^\#, \beta v_2 \right) + \left( (\nabla v_2) v_1^\#, \beta v_2 \right)$$

$$- \frac{1}{\rho \mu} \left[ \left( (\nabla b_1) b_2^\#, \beta v_2 \right) + \left( (\nabla b_2) b_2^\#, \beta v_2 \right) + \left( (\nabla b_2) b_1^\#, \beta v_2 \right) \right]$$

$$- \int_{R^n} (\nabla \beta, v_2) p_2 dx = 0,$$  \hspace{1cm} (5.2)
\[
\frac{1}{2} \frac{d}{dt} \int_{R^n} \beta(x)|b_2|^2 \, dx + \lambda(\nabla b_2, \nabla (\beta b_2)) + ((\nabla b_1)v_2^\beta, \beta b_2) + ((\nabla b_2)v_1^\beta, \beta b_1) \\
- \left[ ((\nabla v_1)b_2^\beta, \beta b_2) + ((\nabla v_2)b_2^\beta, \beta b_2) + ((\nabla v_2)b_1^\beta, \beta b_2) \right] \\
+ \int_{R^n} (\nabla \beta, b_2) q_2 \, dx = 0.
\]

Let us now separately estimate terms on the left-hand sides of (5.2) and (5.3). First, we deal with the terms on the left side of (5.2). For the second term, we have

\[
\nu(\nabla v_2, \nabla (\beta v_2)) \geq \nu \int_{R^n} \beta |\nabla v_2|^2 \, dx - \nu(\nabla v_2, \nabla \beta v_2) \\
\geq \nu \int_{R^n} \beta |\nabla v_2|^2 \, dx - \frac{C}{N} ||\nabla v_2||_2 ||v_2||_2.
\]

For the third term, we apply Hölder’s inequality for exponents, \( r, 2, n \), to get

\[
((\nabla v_1)v_2^\beta, \beta v_2) = -((\nabla (\beta v_2))v_2^\beta, v_1) = -((\nabla \beta v_2)v_2^\beta, v_1) - ((\nabla v_2)v_2^\beta, \beta v_1) \\
\leq \frac{C}{N} ||v_1||_n ||v_2||_r ||v_2^\beta||_2 + ||\beta v_1||_n ||\nabla v_2||_2 ||v_2^\beta||_r
\]

where \( 1/r = (1/2) - (1/n) \). For the fourth term, integration by parts, Hölder’s inequality, and then Sobolov’s inequality yield

\[
|((\nabla v_2)v_2^\beta, \beta v_2)| = \left| \frac{1}{2} ((v_2 \nabla \beta)v_2^\beta, v_2) \right| \\
\leq \frac{C}{N} ||v_2||_2 \left( ||\nabla v_2||_2^2 + ||v_2||^2_2 \right).
\]

For the fifth term, we have

\[
|((\nabla v_2)v_2^\beta, \beta v_2)| \leq ||\beta v_1||_n ||\nabla v_2|| ||v_2||.
\]

The estimates on the sixth and eighth terms can be obtained, respectively, as

\[
\left| \frac{1}{\rho \mu} ((\nabla b_1)b_2^\beta, \beta v_2) \right| \leq \frac{C}{N} ||b_1||_n ||\nabla v_2||_2 ||b_2||_2 + C ||\beta b_1||_n ||\nabla v_2||_2 ||\nabla b_2||_2,
\]

\[
\left| \frac{1}{\rho \mu} ((\nabla b_2)b_2^\beta, \beta v_2) \right| \leq C ||\beta b_1||_n ||\nabla b_2||_2 ||\nabla v_2||_2.
\]

We do not need to estimate the seventh term because it will be canceled with part of the seventh term in (5.3).

Now, let us check terms on the left-hand side of (5.3). Similarly, for the second term, we have

\[
\lambda(\nabla b_2, \nabla (\beta b_2)) \geq \lambda \int_{R^n} \beta |\nabla b_2|^2 \, dx - \frac{C}{N} ||\nabla b_2||_2 ||b_2||_2.
\]

For the third term, we have

\[
((\nabla b_1)v_2^\beta, \beta b_2) \leq \frac{C}{N} ||b_1||_n ||\nabla v_2||_2 ||\nabla b_2||_2 + ||\beta b_1||_n ||\nabla v_2||_2 ||b_2||_2.
\]
For the fourth term, we get
\[
\left| \langle (\nabla b_2) v^p_1, \beta b_2 \rangle \right| \leq \frac{C}{N} \|b_2\|_2 (\|\nabla b_2\|_2^2 + \|b_2\|_2^2).
\] (5.11)

For the fifth and sixth terms, we have, respectively,
\[
\left| \langle (\nabla b_2) v^p_1, \beta b_2 \rangle \right| \leq \|\beta v_1\|_n \|\nabla b_2\|_n \|b_2\|_2
\]
and
\[
\left| \langle (\nabla v_1) b^p_2, \beta b_2 \rangle \right| \leq \frac{C}{N} \|v_1\|_n \|\nabla b_2\|_n \|b_2\|_2 + C \|\beta v_1\|_n \|\nabla b_2\|_2 \|b_2\|_2.
\] (5.12)

The seventh term
\[
\left| - \langle (\nabla v_2) b^p_1, \beta b_2 \rangle - \langle (\nabla b_2) b^p_1, \beta v_2 \rangle \right| \leq \frac{C}{N} \|v_2\|_2 (\|\nabla b_2\|_2^2 + \|b_2\|_2^2).
\] (5.13)

A multiple of the second term on the left-hand side of the above inequality cancels out the seventh term on the left-hand side of (5.2). The estimate on the eighth term can be obtained as
\[
\left| \langle (\nabla v_2) b^p_1, \beta b_2 \rangle \right| \leq C \|\beta b_1\|_n \|\nabla v_2\|_2 \|\nabla b_2\|_2.
\] (5.14)

Now, multiplying (5.3) by 1/ρμ, adding the resulting equation to (5.2), and applying the above estimates yield
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \beta(x) \left( |v_2|^2 + \frac{1}{\rho\mu} |b_2|^2 \right) + \int_{\mathbb{R}^n} \beta(x) \left( v_2 |\nabla v_2|^2 + \frac{1}{\rho\mu} |b_2|^2 \right) dx
\leq \frac{C}{N} \|u_1\|_n \|\nabla u_2\|_2 \|u_2\|_2 + C \|\beta u_1\|_n \|\nabla u_2\|_2 \|\nabla u_2\|_2
\]
\[
+ \int_{\mathbb{R}^n} (\nabla \beta, v_2) p_2 dx + \int_{\mathbb{R}^n} (\nabla \beta, v_2) q_2 dx.
\] (5.15)

We use Riesz transformation to express \(p_2\) and \(q_2\) as
\[
p_2 = R_t R_j \left( v_{11}(v_2)_j^p + v_{21}(v_2)_j^p + v_{21}(v_1)_j^p \right)
- \frac{1}{\rho\mu} b_{11}(b_2)_j^p + b_{21}(b_2)_j^p + b_{21}(b_1)_j^p
\]
and
\[
q_2 = R_t R_j \left( b_{11}(v_2)_j^p + b_{21}(v_2)_j^p + b_{21}(v_1)_j^p - v_{11}(b_2)_j^p - v_{21}(b_2)_j^p - v_{21}(b_1)_j^p \right).
\] (5.16)

Since \(R_t\) is a continuous map from \(L^p\) to itself for \(p > 1\), we have
\[
\left| \int_{\mathbb{R}^n} (\nabla \beta, v_2) p_2 dx + \int_{\mathbb{R}^n} (\nabla \beta, v_2) q_2 dx \right|
\leq \frac{C}{N} \|u_1\|_n \|\nabla u_2\|_2 \|u_2\|_2 + C \|\beta u_1\|_n \|\nabla u_2\|_2 \|\nabla u_2\|_2.
\] (5.17)

Plugging this inequality into (5.15) and integrating over \([0, t]\), we complete the proof of the lemma.

Applying Lemma 5.1 and following the procedure adapted in [1], we can similarly prove the next two lemmas.
Lemma 5.2. (1) Let \( u'_1(0) = (v'_1(0), b'_1(0)) \) be the partition by Lemma 3.1 such that Lemma 4.2 holds. There is a \( T > 0 \), depending only on the norm of \( u'_1(0) \),
\[
\|u'_1(0)\| = \|u'_1(0)\|_n + \|u'_1(0)\|_{n/\beta}, \quad 0 < \beta < 1, \tag{5.18}
\]
such that \( u_1(x,t) \), as a family depending on parameter \( \varepsilon \) and \( 0 < t < T \), is compact in \( L^n \);
(2) the size of \( T \) is determined by
\[
T^{(1-\beta)/2} \left( s^{1-(\beta p/n)} \|u(0)\|_p^{\beta p/n} + s^{1-(\beta p/n)} \|u(0)\|_p^{p/n} \right) < \varepsilon_0; \tag{5.19}
\]
(3) the following inequalities hold
\[
\|u^+_1\|_n \leq c_1(\varepsilon_0) \left( s^{1-(\beta p/n)} \|u(0)\|_p^{\beta p/n} + s^{1-(\beta p/n)} \|u(0)\|_p^{p/n} \right), \tag{5.20}
\]
\[
\|u^+_1\|_p \leq c_2(\varepsilon_0) \left( s^{1-(\beta p/n)} \|u(0)\|_p^{\beta p/n} + \|u^s(0)\|_p \right).
\]

Lemma 5.3. The solution \( u_2(x,t) = (v_2,b_2) \) of (3.12), (3.14), (3.15), and (3.16), as a family depending on the parameter \( \varepsilon \), contains a subfamily that converges in \( L^2 \) of any subset \( S_T \), for \( n = 3,4, T > 0 \).

We are now ready to state and prove the main result of this paper.

Theorem 5.1. Assume that the initial data \((v_0,b_0)\) \( \in L^p(R^n), \) \( 2 < p < n, n = 3,4, \) \( \text{div} \ v_0 = \text{div} \ b_0 = 0. \) Then there exists a weak solution \( u(x,t) = (v(x,t), b(x,t)) \) of (1.1) and (1.2) for all time \( t \), such that, for \( 0 < t < T \), where \( T \) can be arbitrarily large, we have
\[
\|u\|_{2,p} < C, \tag{5.21}
\]
where the constant \( C \) depends on \( T, \|u_0\|_p \).

Proof. From Lemmas 5.2 and 5.3, we have a sequence of solutions \( u_{1m}, u_{2m} \) of (3.11), (3.12), (3.13), (3.14), (3.15), and (3.16) such that \( u_{1m}, u_{2m} \) converge in \( L^n(S_T) \), \( L^2(S_T) \) to \( u_1, u_2 \), respectively. Sending \( m \) to \( \infty \), we see that \( u_1 + u_2 \) is a weak solution of (3.9) for some \( T > 0 \).

By Lemma 5.2, we have
\[
\|u^+_1\|_p \leq c_2(\varepsilon_0) \left( s^{1-(\beta p/n)} \|u(0)\|_p^{\beta p/n} + \|u^s(0)\|_p \right), \quad [0,T]. \tag{5.22}
\]
Fatou’s theorem implies that
\[
\|u^+_1\|_{2,p} \leq T^{1/2} c_2(\varepsilon_0) \left( s^{1-(\beta p/n)} \|u(0)\|_p^{\beta p/n} + \|u^s(0)\|_p \right), \quad [0,T]. \tag{5.23}
\]
Now, for \( u_{2m} \), from a priori estimate for \( u_{2m} \), we have
\[
\|u_{2m}\|_{2,p} \leq c_1(\|\nabla u_{2m}\|_{2,2}, \|u_{2m}\|_2) \leq c_2 \|u_{2m}(0)\|_2 \leq c_2 s^{1-(p/2)} \|u(0)\|^{p/2} \tag{5.24}
\]
Fatou’s theorem implies that
\[
\|u_2\|_{2,p} \leq c_2 s^{1-(p/2)} \|u(0)\|_p^{p/2}. \tag{5.25}
\]
(5.23) and (5.25) implies (5.21).

Due to a priori estimates, we can extend the interval of existence of solution \( u \) from \([0,T]\) to \([T,T_1]\), from \([T,T_1]\) to \([T_1,T_2]\), and so on in such a way that, in each step,
we make sure that $T_{k+1} - T_k > \delta_0$—a fixed constant. Therefore, we obtain the weak solution $u$ for all $t$.

For $n \geq 3$, adapting Calderón’s approach [1], one can also prove the global existence result for system (1.1) and (1.2) as long as the $L^n$ norm of the initial data is suitably small.

References


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