INEQUALITIES VIA CONVEX FUNCTIONS

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ABSTRACT. A general inequality is proved using the definition of convex functions. Many major inequalities are deduced as applications.

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1. Introduction. Kapur and Kumer (1986) have used the principle of dynamical programming to prove major inequalities due to Shannon, Renyi, and Hölder. See [1]. In this note, we prove a general inequality using convex functions. As a result, the inequalities of Shannon, Renyi, Hölder, and others are all deduced.

Let I be an interval in $\mathbb{R}$, $f : I \rightarrow \mathbb{R}$ is said to be convex if and only if, for all $x, y \in I$, all λ , $0 \leq \lambda \leq 1$,

$$f[\lambda x + (1 - \lambda)y] \leq \lambda f(x) + (1 - \lambda)f(y). \quad (1)$$

Here, we give the following new definitions:

- (a) Let f and g be two functions and let I be an interval in $\mathbb{R}$ for which $f \circ g$ is defined, then f is said to be g -convex if and only if, for all $x, y \in I$, all λ , $0 \leq \lambda \leq 1$,

$$f[\lambda g(x) + (1 - \lambda)g(y)] \leq \lambda f \circ g(x) + (1 - \lambda)f \circ g(y). \quad (2)$$

- (b) If the inequality is reversed, then f is said to be g -concave.

If $g(x) = x$, the two definitions of g -convex and convex functions become identical.

THEOREM 1.1. *Let f be g -convex, then*

- (i) *if g is linear, then $f \circ g$ is convex, and*
- (ii) *if f is increasing and g is convex, then $f \circ g$ is convex.*

PROOF.

(i)

$$\begin{aligned} f \circ g[\lambda x + (1 - \lambda)y] &= f[\lambda g(x) + (1 - \lambda)g(y)] \\ &\leq \lambda f \circ g(x) + (1 - \lambda)f \circ g(y). \end{aligned} \quad (3)$$

(ii)

$$\begin{aligned} f \circ g[\lambda x + (1 - \lambda)y] &\leq f[\lambda g(x) + (1 - \lambda)g(y)] \\ &\leq \lambda f \circ g(x) + (1 - \lambda)f \circ g(y). \end{aligned} \quad (4)$$

□

LEMMA 1.1. Let f be g -convex and let $\sum_{i=1}^n t_i = T_n = 1$, $t_i \geq 0$, $i = 1, 2, \dots, n$, then

$$f\left(\sum_{i=1}^n t_i g(x_i)\right) \leq \sum_{i=1}^n t_i f \circ g(x_i). \quad (5)$$

PROOF.

$$\begin{aligned} f\left(\sum_{i=1}^n t_i g(x_i)\right) &= f\left(T_{n-1} \sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} g(x_i) + t_n g(x_n)\right) \\ &\leq T_{n-1} f\left(\sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} g(x_i)\right) + t_n f \circ g(x_n) \\ &= T_{n-2} f\left(\frac{T_{n-2}}{T_{n-1}} \sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} g(x_i) + \frac{t_{n-1}}{T_{n-1}} g(x_{n-1})\right) + t_n f \circ g(x_n) \\ &\leq T_{n-2} f\left(\sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} g(x_i)\right) + t_{n-1} f \circ g(x_{n-1}) + t_n f \circ g(x_n) \\ &\vdots \\ &\leq \sum_{i=1}^n t_i f \circ g(x_i). \end{aligned} \quad (6) \quad \square$$

LEMMA 1.2. For any function g , the exponential function $f(x) = e^x$ is g -convex.

PROOF. Define

$$F(x) = \lambda e^{g(x)} + (1-\lambda)e^{g(y)} - e^{\lambda g(x) + (1-\lambda)g(y)}. \quad (7)$$

Let

$$G(t) = (1-\lambda) + \lambda t - t^\lambda, \quad t > 0. \quad (8)$$

It follows that

$$G'(t) = \lambda(1 - t^{\lambda-1}), \quad G''(t) = \lambda(1-\lambda)t^{\lambda-2}. \quad (9)$$

Thus, $G'(t) = 0$ when $t = 1$ and $G''(1) = \lambda(1-\lambda) > 0$. Hence, G has its minimum value 0 at $t = 1$ and this implies $G(t) \geq 0$, $t > 0$. The result follows by putting $F(x) = e^{g(y)} G(e^{g(x)-g(y)})$. $\square$

COROLLARY 1.3. The function $f(x) = \ln(x)$ is concave for if $h(x) = e^x$, then, by Lemma 1.2, h is f -convex. Hence,

$$e^{\lambda(\ln x) + (1-\lambda)\ln y} \leq \lambda e^{\ln x} + (1-\lambda)e^{\ln y} = \lambda x + (1-\lambda)y. \quad (10)$$

It follows that

$$\lambda \ln x + (1-\lambda) \ln y \leq \ln[\lambda x + (1-\lambda)y]. \quad (11)$$

2. Main inequality

THEOREM 2.1.

$$\sum_{j=1}^n \prod_{i=1}^m (p_{ij})^{q_i / \sum_{i=1}^m q_i} \leq \frac{\sum_{i=1}^m \sum_{j=1}^n p_{ij} q_i}{\sum_{i=1}^m q_i}. \quad (12)$$

PROOF. If $f(x) = e^x$ and $g(x) = \ln x$, then f is g -convex. By Lemma 1.2, we have

$$\begin{aligned}
\prod_{i=1}^m (p_{ij})^{q_i / \sum_{i=1}^m q_i} &= e^{\ln \left(\prod_{i=1}^m (p_{ij})^{q_i / \sum_{i=1}^m q_i} \right)} \\
&= e^{\sum_{i=1}^m \ln(p_{ij})^{q_i / \sum_{i=1}^m q_i}} = e^{\sum_{i=1}^m (q_i / \sum_{i=1}^m q_i) \ln p_{ij}} \\
&\leq \sum_{i=1}^m \left(\frac{q_i}{\sum_{i=1}^m q_i} \right) e^{\ln p_{ij}} = \frac{\sum_{i=1}^m q_i p_{ij}}{\sum_{i=1}^m q_i}.
\end{aligned} \tag{13}$$

Therefore,

$$\sum_{j=1}^n \prod_{i=1}^m (p_{ij})^{q_i / \sum_{i=1}^m q_i} \leq \frac{\sum_{j=1}^n \sum_{i=1}^m p_{ij} q_i}{\sum_{i=1}^m q_i} = \frac{\sum_{i=1}^m \sum_{j=1}^n p_{ij} q_i}{\sum_{i=1}^m q_i}. \tag{14}$$

□

3. Applications

THEOREM 3.1 (Shannon's inequality). *Given $\sum_{i=1}^m a_i = a$, $\sum_{i=1}^m b_i = b$, then*

$$a \ln \left(\frac{a}{b} \right) \leq \sum_{i=1}^m a_i \ln \left(\frac{a_i}{b_i} \right), \quad a_i, b_i \geq 0. \tag{15}$$

PROOF. Applying Theorem 2.1 by putting

$$p_{ij} = \frac{b_i}{a_i}, \quad j = 1, \quad q_i = a_i, \quad \sum_{i=1}^m a_i = a, \quad \sum_{i=1}^m b_i = b, \tag{16}$$

we have

$$\prod_{i=1}^m \left(\frac{b_i}{a_i} \right)^{a_i / \sum_{i=1}^m a_i} \leq \frac{\sum_{i=1}^m b_i}{\sum_{i=1}^m a_i}. \tag{17}$$

That is

$$\prod_{i=1}^m \left(\frac{b_i}{a_i} \right)^{a_i / a} \leq \frac{b}{a}. \tag{18}$$

It follows that

$$\frac{a}{b} \leq \prod_{i=1}^m \left(\frac{a_i}{b_i} \right)^{a_i / a}. \tag{19}$$

Hence, we get

$$a \ln \left(\frac{a}{b} \right) \leq \sum_{i=1}^m a_i \ln \left(\frac{a_i}{b_i} \right). \tag{20}$$

□

THEOREM 3.2 (Renyi's inequality). *Given $\sum_{i=1}^m a_i = a$, $\sum_{i=1}^m b_i = b$, then, for $\alpha > 0$, $\alpha \neq 1$,*

$$\frac{1}{\alpha - 1} (a^\alpha b^{1-\alpha} - a) \leq \sum_{i=1}^m \frac{1}{\alpha - 1} (a_i^\alpha b_i^{1-\alpha} - a_i), \quad a_i, b_i \geq 0. \tag{21}$$

PROOF. Applying Theorem 2.1 with $i = 2$, $p_{1j} = c_j$, $p_{2j} = d_j$, $q_1 = \lambda$, $q_2 = 1 - \lambda$, $0 < \lambda < 1$, we have

$$\sum_{j=1}^m c_j^\lambda d_j^{1-\lambda} \leq \sum_{j=1}^m (\lambda c_j + (1 - \lambda) d_j). \tag{22}$$

On putting $c_j = (a_j / \sum_{j=1}^m a_j)$ and $d_j = (b_j / \sum_{j=1}^m b_j)$, inequality (22) implies

$$\sum_{j=1}^m a_j^\lambda b_j^{1-\lambda} \leq \left(\sum_{j=1}^m a_j \right)^\lambda \left(\sum_{j=1}^m b_j \right)^{1-\lambda}, \quad (23)$$

and this gives

$$\frac{a^\lambda b^{1-\lambda}}{\lambda - 1} \leq \frac{1}{\lambda - 1} \sum_{j=1}^m a_j^\lambda b_j^{1-\lambda}. \quad (24)$$

Thus, for the case $0 < \alpha < 1$, the theorem follows from inequality (24) by setting $\lambda = \alpha$. Now, inequality (23) implies

$$\left(\sum_{j=1}^m a_j^\lambda b_j^{1-\lambda} \right)^{1/\lambda} \left(\sum_{j=1}^m b_j \right)^{1-1/\lambda} \leq \sum_{j=1}^m a_j. \quad (25)$$

Let $a_j^\lambda b_j^{1-\lambda} = e_j$, $\lambda = 1/\alpha$, then inequality (25) gives

$$\frac{1}{\alpha - 1} \left(\sum_{j=1}^m e_j \right)^\alpha \left(\sum_{j=1}^m b_j \right)^{1-\alpha} \leq \frac{1}{\alpha - 1} \sum_{j=1}^m e_j^\alpha b_j^{1-\alpha}. \quad (26)$$

This completes the proof of the theorem. $\square$

THEOREM 3.3 (Generalization of Hölder's inequality).

$$\sum_{j=1}^n \prod_{i=1}^m (p_{ij})^{q_i} \leq \prod_{i=1}^m \left(\sum_{j=1}^n p_{ij} \right)^{q_i}, \quad \sum_{i=1}^m q_i = 1. \quad (27)$$

PROOF. Applying Theorem 2.1 with $p_{ij} / \sum_{j=1}^n p_{ij}$ instead of p_{ij} , we get

$$\sum_{j=1}^n \prod_{i=1}^m \left(\frac{p_{ij}}{\sum_{j=1}^n p_{ij}} \right)^{q_i} \leq \sum_{i=1}^m \left(\sum_{j=1}^n \left(\frac{p_{ij}}{\sum_{j=1}^n p_{ij}} \right) \right)^{q_i} q_i = \sum_{i=1}^m q_i = 1, \quad (28)$$

which implies

$$\sum_{j=1}^n \prod_{i=1}^m (p_{ij})^{q_i} \leq \prod_{i=1}^m \left(\sum_{j=1}^n p_{ij} \right)^{q_i}. \quad (29)$$

$\square$

THEOREM 3.4 (Arithmetic-Geometric-Mean inequality).

$$\left(\prod_{i=1}^m x_i \right)^{1/m} \leq \frac{1}{m} \sum_{i=1}^m x_i. \quad (30)$$

PROOF. Applying Theorem 2.1, with $j = 1$, $p_{ij} = x_i$, $q_i = 1$. $\square$

REFERENCES

- [1] J. N. Kapur, V. Kumar, and U. Kumar, *A measure of mutual divergence among a number of probability distributions*, Internat. J. Math. Math. Sci. **10** (1987), no. 3, 597-607. MR 89d:94030. Zbl 641.94006.

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