ON THE RITT ORDER AND TYPE OF A CERTAIN CLASS OF FUNCTIONS DEFINED BY *BE*-DIRICHLETIAN ELEMENTS

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ABSTRACT. We introduce the notions of Ritt order and type to functions defined by the series

$$\sum_{n=1}^{\infty} f_n(\sigma + i\tau_0) \exp(-s\lambda_n), \quad s = \sigma + i\tau, (\sigma, \tau) \in \mathbf{R} \times \mathbf{R} \quad (*)$$

indexed by τ_0 on $\mathbf{R}$, where $(\lambda_n)_1^\infty$ is a *D*-sequence and $(f_n)_1^\infty$ is a sequence of entire functions of bounded index with at most a finite number of zeros. By definition, the series are *BE*-Dirichletian elements. The notions of order and type of functions, defined by *B*-Dirichletian elements, are considered in [3, 4]. In this paper, using a technique similar to that used by M. Blambert and M. Berland [6], we prove the same properties of Ritt order and type for these functions.

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1. Preliminary lemmas

DEFINITION 1.1 (B. Lepson [10]). An entire function f is said to be of bounded index if there exists a nonnegative integer ν such that

$$\max \left\{ \frac{|f^{(k)}(s)|}{k!} \mid k \in \{0, 1, \dots, \nu\} \right\} \geq \frac{|f^{(j)}(s)|}{j!}, \quad (f^{(0)}(s) = f(s)) \quad (1.1)$$

for all j and for all s . The least such integer ν is called the index of f .

THEOREM A (F. Gross [8]). *An entire function with at most a finite number of zeros is of bounded index if and only if it is of the form $P(s) \exp(\alpha s)$, where $P(s)$ is polynomial and α is a complex constant.*

THEOREM B (S. M. Shah [16]). *Let $f(s) = P(s) \exp(\alpha s)$, where α is any complex number and $P(s)$ is a polynomial of degree less than n . Then f is of bounded index and the index $\nu \leq p$, where p is any integer such that $p \geq n - 1$ and*

$$\frac{n|\alpha|}{p+1} + \left(\frac{n(n-1)}{2!} |\alpha|^2 \right) \frac{1}{p(p+1)} + \dots + \frac{|\alpha|^n}{(p-n+2) \cdots (p+1)} \leq 1. \quad (1.2)$$

Let $(\lambda_n)_1^\infty$ be a *D*-sequence (that is a positive strictly increasing unbounded sequence) and $(f_n)_1^\infty$ be a sequence of entire functions f_n of bounded index ν_n with

at most a finite number of zeros from Theorem A. As a result of the two theorems, we have $\forall s \in \mathbf{C}, \forall n \in \mathbf{N} \setminus \{0\}$

$$f_n(s) = P_n(s) \exp(\alpha_n s), \quad (1.3)$$

where $P_n(s)$ is a polynomial of degree m_n and α_n is a complex constant, that is,

$$s \mapsto P_n(s) = \sum_{j=0}^{m_n} a_{n,j} s^j \quad \text{with } a_{n,m_n} \neq 0 \text{ and } s \in \mathbf{C}. \quad (1.4)$$

Let us suppose that $\exists k \in]0, \lambda_1[, \forall n \in \mathbf{N} \setminus \{0\}$

$$\alpha_n \in \overline{d_{(0,k)}}, \quad (1.5)$$

where $\overline{d_{(0,k)}}$ is the closed disc centered at 0 and of radius k .

Consider the space of elements

$$\{f_{\tau_0}\} : \sum_{n=1}^{\infty} f_n(\sigma + i\tau_0) \exp(-s\lambda_n), \quad s = \sigma + i\tau, (\sigma, \tau) \in \mathbf{R} \times \mathbf{R} \quad (1.6)$$

indexed by τ_0 on $\mathbf{R}$. By definition, $\{f_{\tau_0}\}$ is the BE-Dirichletian element. Let

$$\beta = \limsup_{n \rightarrow \infty} \left\{ \frac{m_n}{\lambda_n} \right\}, \quad (1.7)$$

$$A_n = \max \left\{ |a_{n,j}| \mid j \in \{0, 1, \dots, m_n\} \right\} \quad \forall n \in \mathbf{N} \setminus \{0\}. \quad (1.8)$$

Consider the associated Dirichletian element

$$\{f_A\} : \sum_{n=1}^{\infty} A_n \exp(-s\lambda_n), \quad (1.9)$$

whose coefficients are strictly positive and denote, by $\sigma_c^{f_A}$, the abscissa of convergence of $\{f_A\}$.

Let us state three lemmas due to M. Blambert and M. Berland [6] which we use later. These demonstrations are obvious because this sequence $(\alpha_n)_{n=1}^{\infty}$ is bounded.

LEMMA 1.1. *If $\sigma_c^{f_A} = -\infty, \beta < \infty, \forall n \in \mathbf{N} \setminus \{0\}, \alpha_n \in \overline{d_{(0,k)}}$, then $\{f_{\tau_0}\}$ converges absolutely on $\mathbf{C}$ for any arbitrary τ_0 in $\mathbf{R}$.*

LEMMA 1.2. *If $\sigma_c^{f_A} = -\infty, \beta < \infty, \forall n \in \mathbf{N} \setminus \{0\}, \alpha_n \in \overline{d_{(0,k)}}$, we have $\forall \tau_0 \in \mathbf{R}, \forall \tau \in \mathbf{R}$*

$$\lim_{\sigma \rightarrow -\infty} f_{\tau_0}(\sigma + i\tau) = 0. \quad (1.10)$$

LEMMA 1.3. *If $\sigma_c^{f_A} = -\infty, \forall n \in \mathbf{N} \setminus \{0\}, \alpha_n \in \overline{d_{(0,k)}}$, we have $\forall \tau_0 \in \mathbf{R}, \forall \sigma \in \mathbf{R}$*

$$\begin{aligned} & P_n(\sigma + i\tau_0) \exp[\alpha_n(\sigma + i\tau_0) - \sigma\lambda_n] \\ &= \lim_{\tau_2 \rightarrow \infty} \left\{ \frac{1}{\tau_2} \int_{\tau_1}^{\tau_2} f_{\tau_0}(\sigma + i\tau) \exp(i\tau\lambda_n) d\tau \right\}, \end{aligned} \quad (1.11)$$

where τ_1 is any arbitrary real number.

2. Main theorems. Let us define the following quantities. For each σ on $\mathbf{C}$,

$$M(\sigma; f_{\tau_0}) = \sup \{ |f_{\tau_0}(\sigma' + i\tau')| \mid \sigma' \geq \sigma, \tau' \in \mathbf{R} \}, \quad (2.1)$$

$$M_{n'}(\sigma; f_{\tau_0}) = \sup \{ |f_{\tau_0, n'}(\sigma' + i\tau')| \mid \sigma' \geq \sigma, \tau' \in \mathbf{R} \}, \quad (2.2)$$

$$\mu(\sigma; f_{\tau_0}) = \sup \{ |f_n(\sigma + i\tau_0)| \exp(-\sigma\lambda_n) \mid n \in \mathbf{N} \setminus \{0\} \}, \quad (2.3)$$

$$\mu_{n'}(\sigma; f_{\tau_0}) = \sup \{ |f_n(\sigma + i\tau_0)| \exp(-\sigma\lambda_n) \mid n \geq n' \}; \quad (2.4)$$

where

$$f_{\tau_0, n'}(s) = \sum_{n=n'}^{\infty} f_n(\sigma + i\tau_0) \exp(-s\lambda_n). \quad (2.5)$$

The quantities defined above are finite.

REMARK. The function $\sigma \mapsto M(\sigma; f_{\tau_0})$ is decreasing onto $\mathbf{R}$.

THEOREM 2.1. If $\sigma_c^{f_A} = -\infty$, $\beta < \infty$, $\forall n \in \mathbf{N} \setminus \{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, we have

$$\lim_{\sigma \rightarrow \infty} \{M(\sigma; f_{\tau_0})\} = 0 \quad \text{and} \quad \lim_{\sigma \rightarrow -\infty} \{M(\sigma; f_{\tau_0})\} = \infty. \quad (2.6)$$

PROOF. We have $\forall \varepsilon \in]0, 1[$, $\exists n' (= n'_\varepsilon) \in \mathbf{N} \setminus \{0\}$, $\forall n \geq n'$, $m_n/\lambda_n < \beta + \varepsilon$ and $\exists n'' (= n''_\varepsilon) \in \mathbf{N} \setminus \{0\}$, $\forall n \geq n''$, $k/\lambda_n < \varepsilon$, $\forall \sigma > 0$ such that

$$\begin{aligned} & \sum_{n=n_1 (= \max\{n', n''\})}^{\infty} |f_n(\sigma + i\tau_0)| \exp(-\sigma\lambda_n) \\ & \leq \sum_{n=n_1}^{\infty} A_n \exp \left\{ -\sigma\lambda_n \left[1 - \left(\frac{(\beta + \varepsilon) \log(1 + |\sigma| + |\tau_0|)}{\sigma} + \varepsilon \left(1 + \frac{|\tau_0|}{\sigma} \right) \right) \right] \right\}, \end{aligned} \quad (2.7)$$

$\forall \varepsilon' \in]0, 1 - \varepsilon[$, $\exists \sigma' (= \sigma_{\varepsilon'}) > 0$, $\forall \sigma > \sigma'$,

$$\frac{(\beta + \varepsilon) \log(1 + |\sigma| + |\tau_0|) + \varepsilon|\tau_0|}{\sigma} < \varepsilon' \quad (2.8)$$

and

$$\begin{aligned} & \sigma \left[1 - \left(\frac{(\beta + \varepsilon) \log(1 + |\sigma| + |\tau_0|) + \varepsilon|\tau_0|}{\sigma} + \varepsilon \right) \right] \\ & > \sigma[(1 - \varepsilon) - \varepsilon'] > \sigma'[(1 - \varepsilon) - \varepsilon'] (> 0). \end{aligned} \quad (2.9)$$

Therefore, $\exists n_1 \in \mathbf{N} \setminus \{0\}$ such that

$$M_{n_1}(\sigma; f_{\tau_0}) < f_{A, n_1}(\sigma'[(1 - \varepsilon) - \varepsilon']) = \sum_{n=n_1}^{\infty} A_n \exp[-\sigma'((1 - \varepsilon) - \varepsilon')\lambda_n], \quad (2.10)$$

where

$$\lim_{\sigma \rightarrow \infty} \{M_{n_1}(\sigma; f_{\tau_0})\} = 0. \quad (2.11)$$

On the other hand, we have, $\forall n \in \{1, 2, \dots, n_1 - 1\}$

$$\lim_{\sigma \rightarrow -\infty} \{P_n(\sigma + i\tau_0) \exp[\alpha_n(\sigma + i\tau_0) - \sigma\lambda_n]\} = 0 \quad (\text{with } \lambda_1 > k \text{ and } \alpha_n \in \overline{d_{(0,k)}}). \quad (2.12)$$

We have

$$\lim_{\sigma \rightarrow -\infty} \{M(\sigma; f_{\tau_0})\} = 0. \quad (2.13)$$

On the other hand, $\forall \sigma < 0$, if $M(\sigma; f_{\tau_0})$ is bounded onto $\mathbf{R}$ implies that (from Lemma 1.3)

$$\forall n \in \mathbf{N} \setminus \{0\} : \{j \in \{0, 1, \dots, m_n\} \Rightarrow a_{n,j} = 0\}. \quad (2.14)$$

Or, thus, we get the contradiction that

$$a_{n,m_n} \neq 0 \quad \forall n \in \mathbf{N} \setminus \{0\} \quad (2.15)$$

and

$$\lim_{\sigma \rightarrow -\infty} \{M(\sigma; f_{\tau_0})\} = \infty. \quad (2.16)$$

Thus, (2.13) and (2.16) prove the theorem. $\square$

Furthermore, let

$$\rho_R^{f_{\tau_0}} = \limsup_{\sigma \rightarrow -\infty} \left\{ \frac{\log^+ \left(\log^+ \left(M(\sigma; f_{\tau_0}) \right) \right)}{-\sigma} \right\}, \quad (2.17)$$

$$\lambda_R^{f_{\tau_0}} = \liminf_{\sigma \rightarrow -\infty} \left\{ \frac{\log^+ \left(\log^+ \left(M(\sigma; f_{\tau_0}) \right) \right)}{-\sigma} \right\}. \quad (2.18)$$

By definition, $\rho_R^{f_{\tau_0}}$ and $\lambda_R^{f_{\tau_0}}$ are the Ritt-order and the lower Ritt-order of function f_{τ_0} defined by BE-Dirichletian element $\{f_{\tau_0}\}$. Also, $M(\sigma; f_A)$ is defined in a similar manner with f_A in the place of f_{τ_0} . It is trivial that

$$\rho_R^{f_A} = \limsup_{\sigma \rightarrow -\infty} \left\{ \frac{\log^+ \left(\log^+ (f_A(\sigma)) \right)}{-\sigma} \right\}, \quad (2.19)$$

$$\lambda_R^{f_A} = \liminf_{\sigma \rightarrow -\infty} \left\{ \frac{\log^+ \left(\log^+ (f_A(\sigma)) \right)}{-\sigma} \right\}. \quad (2.20)$$

THEOREM 2.2. *If $\sigma_c^{f_A} = -\infty$, $\beta < \infty$, $\forall n \in \mathbf{N} \setminus \{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, and $L (= \limsup_{n \rightarrow \infty} \{\log n / \lambda_n\}) < \infty$, we have $\forall \tau_0 \in \mathbf{R}$,*

$$\rho_R^{f_{\tau_0}} = \rho_R^{f_A} \quad \text{and} \quad \lambda_R^{f_{\tau_0}} = \lambda_R^{f_A}. \quad (2.21)$$

PROOF. (1) We get the inequalities, $\forall \tau_0 \in \mathbf{R}$,

$$\rho_R^{f_A} \leq \rho_R^{f_{\tau_0}} \quad \text{and} \quad \lambda_R^{f_A} \leq \lambda_R^{f_{\tau_0}}. \quad (2.22)$$

τ_0 is any arbitrary real number. Consider the closed interval $I(s, \lambda) = \{s' \in \mathbf{C} \mid |\sigma' - \sigma| \leq \lambda > 0, \tau' = \tau_0\}$, where $\sigma' = \text{Re}(s')$, $\tau' = \text{Im}(s')$, and $s = \sigma + i\tau_0$. Let $\forall n \in \mathbf{N} \setminus \{0\}$,

$$p_n(s, \lambda) = \sup \left\{ |P_n(s')| \mid s' \in \overline{d_{(s, \lambda)}} \right\}, \quad (2.23)$$

$$p_n^*(s, \lambda) = \sup \left\{ |P_n(s')| \mid s' \in I(s, \lambda) \right\}. \quad (2.24)$$

Using Lemma 1.3, we have $\forall \sigma' \in [\sigma - \lambda, \sigma + \lambda], \forall n \in \mathbf{N} \setminus \{0\}$,

$$|P_n(\sigma' + i\tau_0)| \exp[(\operatorname{Re}(\alpha_n) - \lambda_n)\sigma' - \operatorname{Im}(\alpha_n)\tau_0] \leq M(\sigma - \lambda; f_{\tau_0}), \quad (2.25)$$

and then (M. Blambert and M. Berland [6])

$$p_n^*(s, \lambda) \exp[-(\sigma + \lambda)(\lambda_n - \operatorname{Re}(\alpha_n)) - \operatorname{Im}(\alpha_n)\tau_0] \leq M(\sigma - \lambda; f_{\tau_0}), \quad (2.26)$$

$$6^{-m_n} p_n(s, \lambda) \leq p_n^*(s, \lambda), \quad (2.27)$$

$$A_n(1 + |s|)^{-m_n} \leq p_n(s, \lambda) \quad \forall \lambda \geq 1, \quad (\text{M. Berland [1]}). \quad (2.28)$$

Therefore, we have $\forall \lambda \geq 1, \forall \sigma \in \mathbf{R}, \forall n \in \mathbf{N} \setminus \{0\}$

$$A_n \leq M(\sigma - \lambda; f_{\tau_0}) \exp \left\{ \left[\sigma + \lambda + \frac{m_n}{\lambda'_n} \log(6(1 + |\sigma| + |\tau_0|)) + \frac{\operatorname{Im}(\alpha_n)}{\lambda'_n} \tau_0 \right] \lambda'_n \right\}, \quad (2.29)$$

where $\lambda'_n = \lambda_n - \operatorname{Re}(\alpha_n)$.

We have $\forall \varepsilon \in]0, 1[, \exists n_1 \in \mathbf{N} \setminus \{0\}, \forall n \geq n_1$

$$A_n \leq M(\sigma - \lambda; f_{\tau_0}) \exp \left\{ [\sigma + \lambda + (\beta' + \varepsilon) \log(6(1 + |\sigma| + |\tau_0|)) + \varepsilon] \lambda'_n \right\}, \quad (2.30)$$

and $\forall \varepsilon_1 \in]0, 1[, \exists \sigma_1 (= \sigma_{\varepsilon_1}) > 0, \forall \sigma < -\sigma_1$

$$\frac{\lambda + (\beta' + \varepsilon) \log(6(1 + |\sigma| + |\tau_0|)) + \varepsilon}{-\sigma} < \varepsilon_1, \quad (2.31)$$

where $\beta' = \limsup_{n \rightarrow \infty} \{m_n / \lambda'_n\} (< \infty)$ which implies that, $\forall \varepsilon_1 \in]0, 1[, \forall n \geq n_1, \forall \sigma < \sigma_1$

$$A_n \leq M(\sigma - \lambda; f_{\tau_0}) \exp[\sigma(1 - \varepsilon_1)\lambda'_n]. \quad (2.32)$$

Now, $\forall n \in \mathbf{N} \setminus \{0\}, \operatorname{Re}(\alpha_n) \leq k$ (because $\alpha_n \in \overline{d_{(0, k)}}$)

$$\lambda'_n = \lambda_n \left(1 - \frac{\operatorname{Re}(\alpha_n)}{\lambda_n} \right) \geq \lambda_n \left(1 - \frac{k}{\lambda_n} \right). \quad (2.33)$$

We have, $\forall \varepsilon_2 \in]0, 1[, \exists n' \in \mathbf{N} \setminus \{0\}, \forall n \geq n'$,

$$\frac{k}{\lambda_n} < \varepsilon_2 \quad \text{and} \quad \lambda'_n \geq \lambda_n(1 - \varepsilon_2) \quad (\Rightarrow \beta' = \beta) \quad (2.34)$$

which implies that, $\forall \lambda \geq 1, \forall n \geq \max\{n_1, n'\} (= n_2), \forall \sigma < -\sigma_1$

$$A_n \exp[-\sigma(1 - \varepsilon_1)(1 - \varepsilon_2)\lambda_n] \leq M(\sigma - \lambda; f_{\tau_0}). \quad (2.35)$$

Put, $\forall n \in \mathbf{N} \setminus \{0\}$

$$\mu_{n(1,2)} = (1 - \varepsilon_1)(1 - \varepsilon_2)\lambda_n \quad (> 0), \quad (2.36)$$

$(\mu_{n(1,2)})$ is a D -sequence. Consider the Dirichletian element

$$\{f_{A(1,2)}\} : \sum_{n=1}^{\infty} A_n \exp(-s\mu_{n(1,2)}) \quad (2.37)$$

indexed by the couple $(1,2)$ and denote, by $\sigma_c^{f_{A(1,2)}}$, the abscissa of convergence of $\{f_{A(1,2)}\}$. We have, $\sigma_c^{f_{A(1,2)}} = -\infty$ (since $\sigma_c^{f_A} = -\infty$), $f_{A(1,2)}$ is an entire function and, its Ritt-order is

$$\rho_R^{f_{A(1,2)}} = \limsup_{\sigma \rightarrow -\infty} \left\{ \frac{\log^+ \left(\log^+ \left(f_{A(1,2)}(\sigma) \right) \right)}{-\sigma} \right\}, \quad (2.38)$$

and its lower Ritt-order is

$$\lambda_R^{f_{A(1,2)}} = \liminf_{\sigma \rightarrow -\infty} \left\{ \frac{\log^+ \left(\log^+ \left(f_{A(1,2)}(\sigma) \right) \right)}{-\sigma} \right\}. \quad (2.39)$$

Now, $\forall \sigma \in \mathbf{R}$,

$$f_{A(1,2)}(\sigma) = f_A(\sigma(1-\varepsilon_1)(1-\varepsilon_2)) \quad (2.40)$$

and

$$\rho_R^{f_{A(1,2)}} = (1-\varepsilon_1)(1-\varepsilon_2)\rho_R^{f_A}, \quad (2.41)$$

$$\lambda_R^{f_{A(1,2)}} = (1-\varepsilon_1)(1-\varepsilon_2)\lambda_R^{f_A}. \quad (2.42)$$

Put (Q. S. Liu [11]) $\forall \sigma \in \mathbf{R}$,

$$\mu_{n_2}(\sigma; f_{A(1,2)}) = \sup \left\{ A_n \exp(-\sigma\mu_{n(1,2)}) \mid n \geq n_2 \right\}. \quad (2.43)$$

We have $\forall \varepsilon > 0, \forall \sigma \in \mathbf{R}$

$$\begin{aligned} f_{A(1,2),n_2}(\sigma) &\leq \sum_{n=n_2}^{\infty} \left(A_n \exp[-(\sigma-L-\varepsilon)\mu_{n(1,2)}] \right) \exp[-(L+\varepsilon)\mu_{n(1,2)}] \\ &\leq \mu_{n_2}(\sigma-L-\varepsilon; f_{A(1,2)}) K_{n_2}(\varepsilon) \end{aligned} \quad (2.44)$$

with $K_{n_2}(\varepsilon) = \sum_{n=n_2}^{\infty} \exp[-(L+\varepsilon)\mu_{n(1,2)}]$.

Hence, $\forall \varepsilon > 0, \forall \sigma \in \mathbf{R}$

$$f_{A(1,2),n_2}(\sigma) \leq \mu_{n_2}(\sigma-L-\varepsilon; f_{A(1,2)}) K_{n_2}(\varepsilon). \quad (2.45)$$

Then we have, $\forall \lambda \geq 1, \exists n_2 = \max\{n_1, n'\}, \forall n \geq n_2, \forall \sigma < -\sigma_1$

$$A_n \exp(-\sigma\mu_{n(1,2)}) \leq M(\sigma-\lambda; f_{\tau_0}). \quad (2.46)$$

This implies that

$$\mu_{n_2}(\sigma; f_{A(1,2)}) \leq M(\sigma-\lambda; f_{\tau_0}). \quad (2.47)$$

From (2.45) and (2.47), we have

$$f_{A(1,2),n_2}(\sigma+L+\varepsilon) \leq M(\sigma-\lambda; f_{\tau_0}) K_{n_2}(\varepsilon), \quad (2.48)$$

$$\rho_R^{f_{A(1,2),n_2}} \leq \rho_R^{f_{\tau_0}} \left(\lim_{\sigma \rightarrow -\infty} \left(\frac{\sigma-\lambda}{\sigma+L+\varepsilon} \right) \right) = \rho_R^{f_{\tau_0}}. \quad (2.49)$$

Or

$$\rho_R^{f_{A(1,2),n_2}} = \rho_R^{f_{A(1,2)}} \quad (\text{M. Blambert [5]}). \quad (2.50)$$

We have, $\forall \varepsilon_1 \in]0, 1[, \forall \varepsilon_2 \in]0, 1[$

$$\rho_R^{f_{A(1,2)}} = (1 - \varepsilon_1)(1 - \varepsilon_2)\rho_R^{f_A} \leq \rho_R^{f_{\tau_0}}, \quad (2.51)$$

$$\lambda_R^{f_{A(1,2)}} = (1 - \varepsilon_1)(1 - \varepsilon_2)\lambda_R^{f_A} \leq \lambda_R^{f_{\tau_0}}. \quad (2.52)$$

As ε_1 and ε_2 are arbitrary, we have

$$\rho_R^{f_A} \leq \rho_R^{f_{\tau_0}}, \quad \text{and then} \quad \lambda_R^{f_A} \leq \lambda_R^{f_{\tau_0}}. \quad (2.53)$$

(2) We get the inequalities, $\forall \tau_0 \in \mathbf{R}$,

$$\rho_R^{f_{\tau_0}} \leq \rho_R^{f_A} \quad \text{and} \quad \lambda_R^{f_{\tau_0}} \leq \lambda_R^{f_A}. \quad (2.54)$$

From Theorem 2.1, we have $\forall \varepsilon \in]0, 1[, \forall \varepsilon' \in]0, 1 - \varepsilon[, \exists \sigma' > 0, \forall \sigma < -\sigma'$

$$\begin{aligned} & \sigma \left[1 - \left((\beta + \varepsilon) \frac{\log(1 + |\sigma| + |\tau_0|)}{|\sigma|} + \varepsilon \left(1 + \frac{|\tau_0|}{|\sigma|} \right) \right) \theta_\sigma \right] \\ &= \sigma \left[1 + (\beta + \varepsilon) \frac{\log(1 + |\sigma| + |\tau_0|)}{|\sigma|} + \varepsilon \left(1 + \frac{|\tau_0|}{|\sigma|} \right) \right] \\ &> \sigma \left[1 + ((\beta + 2\varepsilon)\varepsilon' + \varepsilon) \right] = \sigma(1 + \varepsilon_1), \end{aligned} \quad (2.55)$$

where $\varepsilon_1 = (\beta + 2\varepsilon)\varepsilon' + \varepsilon$, $\theta_\sigma = 1$ if $\sigma > 0$ and $\theta_\sigma = -1$ if $\sigma < 0$.

Hence, $\forall \sigma < -\sigma', \exists n_1 \in \mathbf{N} \setminus \{0\}$,

$$M_{n_1}(\sigma; f_{\tau_0}) \leq f_{A, n_1}(\sigma(1 + \varepsilon_1)), \quad (2.56)$$

which implies that

$$\rho_R^{f_{\tau_0, n_1}} \leq \rho_R^{f_{A, n_1}}(1 + \varepsilon_1) = (1 + \varepsilon_1)\rho_R^{f_A}, \quad (2.57)$$

and where

$$\rho_R^{f_{\tau_0, n_1}} \leq \rho_R^{f_A} \quad \text{and} \quad \lambda_R^{f_{\tau_0, n_1}} \leq \lambda_R^{f_A}. \quad (2.58)$$

Now, $\sigma \in \mathbf{R}$,

$$M(\sigma; f_{\tau_0}) \leq M_{n_1}^0(\sigma; f_{\tau_0}) + M_{n_1}(\sigma; f_{\tau_0}), \quad (2.59)$$

where

$$M_{n_1}^0(\sigma; f_{\tau_0}) = \sup \left\{ |f_{\tau_0, n_1}^0(\sigma' + i\tau')| \mid \sigma' \geq \sigma, \tau' \in \mathbf{R} \right\} \quad (2.60)$$

and

$$\left\{ f_{\tau_0, n_1}^0 \right\} : \sum_{n=1}^{n_1-1} f_n(\sigma + i\tau_0) \exp(-s\lambda_n). \quad (2.61)$$

Then $\forall \tau_0 \in \mathbf{R}$,

$$\rho_R^{f_{\tau_0}} \leq \max \left\{ \rho_R^{f_{\tau_0, n_1}}, \rho_R^{f_{\tau_0, n_1}^0} \right\} = \rho_R^{f_{\tau_0, n_1}} \quad (2.62)$$

since $\rho_R^{f_{\tau_0, n_1}^0} = 0$.

Finally, we have

$$\rho_R^{f_{\tau_0}} \leq \rho_R^{f_A}, \quad (2.63)$$

and, similarly, we can show that

$$\lambda_R^{f_{\tau_0}} \leq \lambda_R^{f_A}. \quad (2.64)$$

Hence, (1) and (2) implies (2.21) which proves this theorem. $\square$

If $\rho_R^{f_{\tau_0}} > 0$, we put

$$\tau_R^{f_{\tau_0}} = \limsup_{\sigma \rightarrow -\infty} \left\{ \frac{\log(M(\sigma; f_{\tau_0}))}{\exp(-\sigma \rho_R^{f_{\tau_0}})} \right\}, \quad (2.65)$$

$$\nu_R^{f_{\tau_0}} = \liminf_{\sigma \rightarrow -\infty} \left\{ \frac{\log(M(\sigma; f_{\tau_0}))}{\exp(-\sigma \rho_R^{f_{\tau_0}})} \right\}. \quad (2.66)$$

By definition, $\tau_R^{f_{\tau_0}}$ and $\nu_R^{f_{\tau_0}}$ are the Ritt-type and the lower Ritt-type of order of f_{τ_0} .

It is trivial that if $\rho_R^{f_A} > 0$,

$$\tau_R^{f_A} = \limsup_{\sigma \rightarrow -\infty} \left\{ \frac{\log(f_A(\sigma))}{\exp(-\sigma \rho_R^{f_A})} \right\}, \quad (2.67)$$

$$\nu_R^{f_A} = \liminf_{\sigma \rightarrow -\infty} \left\{ \frac{\log(f_A(\sigma))}{\exp(-\sigma \rho_R^{f_A})} \right\}. \quad (2.68)$$

THEOREM 2.3. *If $\sigma_c^{f_A} = -\infty$, $\beta < \infty$, $L(= \lim_{n \rightarrow \infty} (\log n / \lambda_n)) = 0$, $\forall n \in \mathbb{N} \setminus \{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, $\rho_R^{f_{\tau_0}} > 0$, we have $\forall \tau_0 \in \mathbb{R}$,*

$$\tau_R^{f_{\tau_0}} = \tau_R^{f_A}. \quad (2.69)$$

PROOF. (1) We have the inequality, $\forall \tau_0 \in \mathbb{R}$,

$$\tau_R^{f_A} \leq \tau_R^{f_{\tau_0}}, \quad (2.70)$$

τ_0 is any arbitrary real number. From Theorem 2.2, we have, $\forall \varepsilon \in]0, 1[$, $\exists n_1 \in \mathbb{N} \setminus \{0\}$, $\forall n \geq n_1$, $\forall \sigma \in \mathbb{R}$; $\forall \lambda \geq 1$,

$$A_n \leq M(\sigma - \lambda; f_{\tau_0}) \exp \left\{ [\sigma + \lambda + (\beta' + \varepsilon) \log(6(1 + |\sigma| + |\tau_0|))] + \varepsilon \right\} \lambda'_n, \quad (2.71)$$

where

$$\lambda'_n = \lambda_n - \operatorname{Re}(\alpha_n) \quad \text{and} \quad \beta' = \limsup_{n \rightarrow \infty} \left\{ \frac{m_n}{\lambda'_n} \right\} \quad (\beta' = \beta) < \infty. \quad (2.72)$$

Now, $\forall \sigma < 0$,

$$A_n \leq M(\sigma - \lambda; f_{\tau_0}) \exp \left\{ \left[\sigma - \lambda - \frac{\sigma(2\lambda + (\beta + \varepsilon) \log(6(1 + |\sigma| + |\tau_0|))) + \varepsilon}{-\sigma} \right] \lambda'_n \right\}. \quad (2.73)$$

Also, we have $\forall \varepsilon_1 \in]0, 1[, \exists \sigma_1 (= \sigma_{\varepsilon_1}) > 0, \forall \sigma < -\sigma_1,$

$$\frac{2\lambda(\beta + \varepsilon) \log(6(1 + |\sigma| + |\tau_0|)) + \varepsilon}{-\sigma} < \varepsilon_1, \quad (2.74)$$

$$\begin{aligned} A_n &\leq M(\sigma - \lambda; f_{\tau_0}) \exp \left\{ [\sigma(1 - \varepsilon_1) - \lambda] \lambda'_n \right\} \\ &\leq M(\sigma - \lambda; f_{\tau_0}) \exp \left[\left(\sigma - \frac{\lambda}{1 - \varepsilon_1} \right) \mu_{n(1,2)} \right], \end{aligned} \quad (2.75)$$

where $\forall n \geq n_2 = \max \{n_1, n\},$

$$\mu_{n(1,2)} = (1 - \varepsilon_1)(1 - \varepsilon_2)\lambda_n, \quad \lambda'_n \geq \lambda_n(1 - \varepsilon_2) \quad (\varepsilon_2 \in]0, 1[) \quad (2.76)$$

which implies that, $\forall \lambda \geq 1, \forall n \geq n_2, \forall \sigma < -\sigma_1$

$$A_n \leq M(\sigma - \lambda; f_{\tau_0}) \exp \left[\sigma \left(1 - \frac{\lambda}{1 - \varepsilon_1} \right) \mu_{n(1,2)} \right]. \quad (2.77)$$

$\tau_R^{f_{\tau_0}}$ is the Ritt-type of order of f_{τ_0} , we have, $\forall \varepsilon' > 0, \exists \sigma' (= \sigma_{\varepsilon'}) > 0, \forall \sigma < -\sigma'$

$$\log(M(\sigma - \lambda; f_{\tau_0})) \leq (\tau_R^{f_{\tau_0}} + \varepsilon') \exp \left[-(\sigma - \lambda) \rho_R^{f_{\tau_0}} \right]. \quad (2.78)$$

Hence, $\forall n \geq n_2, \forall \varepsilon' > 0,$

$$\log A_n \leq (\tau_R^{f_{\tau_0}} + \varepsilon') \exp \left[-(\sigma - \lambda) \rho_R^{f_{\tau_0}} \right] + \left(\sigma - \frac{\lambda}{(1 - \varepsilon_1)} \right) \mu_{n(1,2)}. \quad (2.79)$$

Let us consider f_n , the function defined by

$$f_n(\sigma) = a \exp [-(\sigma - \lambda)b] + \mu_{n(1,2)}(\sigma + c), \quad (2.80)$$

and indexed by $n > n_2$. Choosing

$$a = \tau_R^{f_{\tau_0}} + \varepsilon' > 0, \quad b = \rho_R^{f_{\tau_0}} > 0, \quad c = \frac{\lambda}{\varepsilon_1 - 1}, \quad (2.81)$$

we get $\forall n \geq n_2, \forall \sigma \in \mathbb{R} \setminus \{\sigma_n\},$

$$f_n(\sigma) > f_n(\sigma_n) \quad (2.82)$$

with

$$\sigma_n - \lambda = \frac{1}{b} \log \left(\frac{ab}{\mu_{n(1,2)}} \right) \quad (2.83)$$

and

$$\lim_{n \rightarrow \infty} \sigma_n = -\infty \implies \exists n_3 \in \mathbb{N} \setminus \{0\}, \quad \forall n \geq \max \{n_2, n_3\}, \quad (2.84)$$

$$\sigma_n < -\max \{\sigma_1, \sigma'\}, \quad (2.85)$$

where

$$\begin{aligned} \log A_n \leq f_n(\sigma_n) &= \frac{\mu_{n(1,2)}}{b} + \left(\frac{\varepsilon_1 \lambda}{1 - \varepsilon_1} + \frac{1}{b} \log \frac{ab}{\mu_{n(1,2)}} \right) \mu_{n(1,2)} \\ &\Updownarrow \\ \mu_{n(1,2)} \left(A_n^{b/\mu_{n(1,2)}} \right) &\leq eab \exp \left(\frac{\varepsilon_1 \lambda b}{1 - \varepsilon_1} \right). \end{aligned} \quad (2.86)$$

Or, if $L(= \lim_{n \rightarrow \infty} (\log n / \lambda_n)) = 0$, we have

$$\tau_R^{f_{A(1,2)}} e \rho_R^{f_{A(1,2)}} = \limsup_{n \rightarrow \infty} \left\{ \mu_{n(1,2)} \left(A_n^{\rho_R^{f_{A(1,2)}} / \mu_{n(1,2)}} \right) \right\}, \quad (2.87)$$

(M. Berland [3], following the theorem of Lindelöf-Blambert-Yu) and

$$\rho_R^{f_{\tau_0}} = \rho_R^{f_A} = \frac{1}{(1 - \varepsilon_1)(1 - \varepsilon_2)} \rho_R^{f_{A(1,2)}}, \quad (\text{Theorem 2.2}), \quad (2.88)$$

$$\tau_R^{f_A} = \tau_R^{f_{A(1,2)}}, \quad (2.89)$$

from which

$$(1 - \varepsilon_1)(1 - \varepsilon_2) \lambda_n \left(A_n^{\rho_R^{f_A} / [(1 - \varepsilon_1)(1 - \varepsilon_2) \lambda_n]} \right) \leq e \left(\tau_R^{f_{\tau_0}} + \varepsilon' \right) \rho_R^{f_A} \exp \left(\frac{\varepsilon_1 \lambda \rho_R^{f_{\tau_0}}}{1 - \varepsilon_1} \right) \quad (2.90)$$

and

$$A_n^{\rho_R^{f_A} / \lambda_n} \leq A_n^{\rho_R^{f_A} / [(1 - \varepsilon_1)(1 - \varepsilon_2) \lambda_n]}. \quad (2.91)$$

Then, $\forall \varepsilon_1 \in]0, 1[, \forall \varepsilon_2 \in]0, 1[, \forall \varepsilon' > 0, \rho_R^{f_A} > 0$,

$$\tau_R^{f_A} \leq \frac{\tau_R^{f_{\tau_0}} + \varepsilon'}{(1 - \varepsilon_1)(1 - \varepsilon_2)} \exp \left(\frac{\varepsilon_1 \lambda \rho_R^{f_{\tau_0}}}{1 - \varepsilon_1} \right), \quad (2.92)$$

as $\varepsilon_1, \varepsilon_2$, and ε' are arbitrary, we deduce immediately that

$$\forall \tau_0 \in \mathbf{R} : \tau_R^{f_A} \leq \tau_R^{f_{\tau_0}}. \quad (2.93)$$

(2) We get, when τ_0 is a fixed real number, $\forall \varepsilon > 0, \exists \sigma_\varepsilon > 0, \forall \sigma < -\sigma_\varepsilon$,

$$|f_{\tau_0}(\sigma + i\tau)| \leq |f_A(\sigma(1 + \varepsilon))|. \quad (2.94)$$

In particular, $\forall \varepsilon' > 0, \exists \sigma_{\varepsilon'} > 0, \forall \varepsilon \in]0, \varepsilon' / \sigma_{\varepsilon'}[, \exists \sigma_\varepsilon > 0, \forall \sigma < -\max\{\sigma_\varepsilon, \sigma_{\varepsilon'}\}$

$$f_A(\sigma(1 + \varepsilon)) \leq f_A(\sigma - \varepsilon'), \quad (\text{M. Berland [3]}) \quad (2.95)$$

and, hence, $\forall \varepsilon' > 0$,

$$\forall \sigma < -\max\{\sigma_\varepsilon, \sigma_{\varepsilon'}\} : M(\sigma; f_{\tau_0}) \leq f_A(\sigma - \varepsilon'). \quad (2.96)$$

From $\rho_R^{f\tau_0} = \rho_R^{fA} > 0$, we get the inequality, $\forall \varepsilon' > 0$,

$$\tau_R^{f\tau_0} \leq \tau_R^{fA} \exp(\varepsilon' \rho_R^{fA}). \quad (2.97)$$

As ε' is arbitrary, we have

$$\forall \tau_0 \in \mathbf{R} : \tau_R^{f\tau_0} \leq \tau_R^{fA}. \quad (2.98)$$

As a result of this theorem, we have an expression for $\tau_R^{f\tau_0}$ in terms of λ_n and A_n .

If $\sigma_c^{fA} = -\infty$, $\beta < \infty$, $\forall n \in \mathbf{N} \setminus \{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, $\forall \tau_0 \in \mathbf{R}$,

$$L\left(\lim_{n \rightarrow \infty} \left(\frac{\log n}{\lambda_n}\right)\right) = 0, \quad \rho_R^{f\tau_0} > 0, \quad (2.99)$$

we have

$$\tau_R^{f\tau_0} e \rho_R^{f\tau_0} = \limsup_{n \rightarrow \infty} \left\{ \lambda_n \left(A_n^{\rho_R^{f\tau_0} / \lambda_n} \right) \right\}. \quad (2.100)$$

□

REMARK. The notions of Ritt-type of order of functions, defined by B -Dirichletian elements, are considered in [3] with the same result of this theorem.

THEOREM 2.4. If $\sigma_c^{fA} = -\infty$, $\beta < \infty$, $\forall n \in \mathbf{N} \setminus \{0\}$, $\alpha_n \in \overline{d_{(0,k)}}$, $L = 0$, $\lambda_n \sim \lambda_{n+1}$, φ defined by

$$\varphi(n) = \frac{\log(A_n / A_{n+1})}{\lambda_{n+1} - \lambda_n}, \quad (2.101)$$

is a nondecreasing function of $n \geq n_1$, and $\rho_R^{fA} > 0$, we have, $\forall \tau_0 \in \mathbf{R}$,

$$\nu_R^{f\tau_0} = \nu_R^{fA}. \quad (2.102)$$

PROOF. (1) We have the inequality, $\forall \tau_0 \in \mathbf{R}$,

$$\nu_R^{fA} \leq \nu_R^{f\tau_0}. \quad (2.103)$$

Suppose that the inequality is false. Then

$$\exists \tau_0 \in \mathbf{R} : \nu_R^{f\tau_0} < \nu_R^{fA}. \quad (2.104)$$

Let $\varepsilon \in]0, \nu_R^{fA} - \nu_R^{f\tau_0}[$, $\varepsilon' \in]0, \varepsilon / \nu_R^{fA}[$ and $\nu = \nu_R^{fA-\varepsilon} / (1 - \varepsilon')$; then $\nu_R^{f\tau_0} < \nu < \nu_R^{fA}$. Under the conditions stated in Theorem 2.4, R. K. Srivastava [17] proved that

$$\nu_R^{fA} e \rho_R^{fA} = \liminf_{n \rightarrow \infty} \left\{ \lambda_n \left(A_n^{\rho_R^{fA} / \lambda_n} \right) \right\}, \quad (2.105)$$

which implies that $\exists n' \in \mathbf{N} \setminus \{0\}$, $\forall n \geq n'$,

$$\nu e \rho_R^{fA} < \lambda_n \left(A_n^{\rho_R^{fA} / \lambda_n} \right). \quad (2.106)$$

Now, $\forall \varepsilon_1 \in]0, 1[, \forall \varepsilon_2 \in]0, 1[, \exists \sigma_1 (= \sigma_{\varepsilon_1}) > 0, \forall \sigma < -\sigma_1, \forall n \geq n_2 (= \max\{n_1, n'\})$,

$$A_n \leq M(\sigma - \lambda; f_{\tau_0}) \exp \left[\left(\sigma - \frac{\lambda}{1 - \varepsilon_1} \right) \mu_{n(1,2)} \right], \quad (2.107)$$

where λ is a constant lying in $[1, \infty[$ and

$$\mu_{n(1,2)} = (1 - \varepsilon_1)(1 - \varepsilon_2)\lambda_n \quad (\text{see Theorem 2.3}), \quad (2.108)$$

which gives

$$\log A_n - \left(\sigma - \frac{\lambda}{1 - \varepsilon_1} \right) \mu_{n(1,2)} \leq \log \left(M(\sigma - \lambda; f_{\tau_0}) \right), \quad (2.109)$$

$$\log A_n + \frac{\lambda}{1 - \varepsilon_1} \mu_{n(1,2)} - \sigma \mu_{n(1,2)} \leq \log \left(M(\sigma - \lambda; f_{\tau_0}) \right). \quad (2.110)$$

Let us consider φ_n , the function defined by

$$\varphi_n(\sigma) = \frac{\alpha_n - \beta_n \sigma}{\exp[-(\sigma - \lambda)\rho_R^{f_A}]}, \quad (2.111)$$

and indexed by $n \geq n_2$. Choose

$$\alpha_n = \log A_n + \frac{\lambda}{1 - \varepsilon_1} \mu_{n(1,2)}, \quad \beta_n = \mu_{n(1,2)} \quad (> 0). \quad (2.112)$$

This takes the maximum value at

$$\sigma_n = \frac{\alpha_n}{\beta_n} - \frac{1}{\rho_R^{f_A}} \quad \left(= \frac{\log A_n}{\mu_{n(1,2)}} + \frac{\lambda}{1 - \varepsilon_1} - \frac{1}{\rho_R^{f_A}} \right), \quad (2.113)$$

$$\begin{aligned} \max \{ \varphi_n(\sigma) \mid \sigma \in \mathbf{R} \} &= \frac{\mu_{n(1,2)}}{\rho_R^{f_A} e} \left(A^{\rho_R^{f_A} / \mu_{n(1,2)}} \right) \exp \left(\frac{\varepsilon_1 \lambda}{1 - \varepsilon_1} \rho_R^{f_A} \right) \\ &\leq \frac{\log M(\sigma_n - \lambda; f_{\tau_0})}{\exp[-(\sigma_n - \lambda)\rho_R^{f_{\tau_0}}]} \quad (\text{for } \rho_R^{f_A} = \rho_R^{f_{\tau_0}}). \end{aligned} \quad (2.114)$$

As $\forall n \in \mathbf{N} \setminus \{0\}$,

$$\mu_{n(1,2)} < \lambda_n \iff A^{\rho_R^{f_A} / \mu_{n(1,2)}} > A_n^{\rho_R^{f_A} / \lambda_n} \quad (2.115)$$

which gives

$$\frac{(1 - \varepsilon_1)(1 - \varepsilon_2)}{\rho_R^{f_A} e} \exp \left(\frac{\varepsilon_1 \lambda}{1 - \varepsilon_1} \rho_R^{f_A} \right) \lambda_n \left(A_n^{\rho_R^{f_A} / \lambda_n} \right) \leq \frac{\log M(\sigma_n - \lambda; f_{\tau_0})}{\exp[-(\sigma_n - \lambda)\rho_R^{f_{\tau_0}}]}. \quad (2.116)$$

Finally, we have, $\forall \varepsilon_3 > 0, \exists (\sigma_n)_1^\infty, \lim_{n \rightarrow \infty} \sigma_n = -\infty$,

$$\frac{\log M(\sigma_n - \lambda; f_{\tau_0})}{\exp[-(\sigma_n - \lambda)\rho_R^{f_{\tau_0}}]} \leq v_R^{f_{\tau_0}} + \varepsilon_3. \quad (2.117)$$

Hence, we get, $\forall \varepsilon_3 > 0, \forall \varepsilon_1 \in]0, 1[, \forall \varepsilon_2 \in]0, 1[$,

$$\left((1 - \varepsilon_1)(1 - \varepsilon_2) \exp \left(\frac{\varepsilon_1 \lambda}{1 - \varepsilon_1} \rho_R^{f_A} \right) \right) v \leq v_R^{f_{\tau_0}} + \varepsilon_3. \quad (2.118)$$

Choosing $\varepsilon_3 = \varepsilon_1 = \varepsilon_2$ of $]0, 1[$, we get

$$v \leq v_R^{f_{\tau_0}}. \quad (2.119)$$

Thus, we get the contradiction that

$$v_R^{f_{\tau_0}} < (v \leq) v_R^{f_{\tau_0}} \quad (2.120)$$

which proves, under the stated conditions, that it is impossible to find a τ_0 of $\mathbf{R}$ such that $v_R^{f_{\tau_0}} < v_R^{f_A}$.

(2) We have the inequality, $\forall \tau_0 \in \mathbf{R}$,

$$v_R^{f_{\tau_0}} \leq v_R^{f_A} \quad (\text{see Theorem 2.3, 2}). \quad (2.121)$$

As a result of this theorem, we have an expression for $v_R^{f_{\tau_0}}$ in terms of λ_n and A_n , $\forall \tau_0 \in \mathbf{R}$,

$$v_R^{f_{\tau_0}} e^{\rho_R^{f_{\tau_0}}} = \liminf_{n \rightarrow \infty} \left\{ \lambda_n \left(A_n^{\rho_R^{f_{\tau_0}} / \lambda_n} \right) \right\}. \quad (2.122)$$

□

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