LIMIT SETS IN PRODUCT OF SEMI-DYNAMICAL SYSTEMS

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ABSTRACT. Continuing the study of the properties of Poisson stability and distality [4], we mention the conditions under which \( \Omega_x(x) = \Pi \Omega_\alpha(x_\alpha), \alpha \in I \) and thus, the product of Poisson stable motions remains Poisson stable in the product system.

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1. Introduction. We deal mainly with the product of \( w \)-limit sets in the product space of semi-dynamical systems (s.d.s). In [1], Prem Bajaj has shown that the product of semi-dynamical systems is a semi-dynamical system. He has also shown that \( \Pi \Omega_\alpha(x_\alpha), \alpha \in I \) contains the \( w \)-limit set \( \Omega_x(x) \) of \( x \) in the product system. In general, equality does not hold in the above. Indeed \( \Omega_x(x) \) may be empty. He has given two theorems: one in which \( \Omega_x(x) \) is nonempty and the other indicating a case of equality viz. Theorems 2.3 and 2.4.

In this paper, continuing the study of the properties of Poisson stability and distality [4], we mention the conditions under which \( \Omega_x(x) = \Pi \Omega_\alpha(x_\alpha), \alpha \in I, x = \{x_\alpha\} \) and therefore, the product of Poisson stable motions, under these conditions, is Poisson stable.

2. Definitions and notations

**Definition 2.1.** A continuous mapping \( \pi : X \times \mathbb{R}^+ \rightarrow X \) on a topological space \( X \) is said to define a semi-dynamical system \( (X, \pi) \) if \( \pi(x, 0) = x \) and \( \pi(\pi(x, t), s) = \pi(x, t + s) \) for every \( x \in X \) and \( t, s \in \mathbb{R}^+ \). (\( \mathbb{R}^+ \) denotes the set of nonnegative reals.)

**Definition 2.2.** Let \( (X_\alpha, \pi_\alpha), \alpha \in I \) be a family of dynamical systems. Let \( X = \Pi X_\alpha \) be the product space. Let \( x \in X \) and \( x = \{x_\alpha\} \). Define a map \( \pi \) from \( X \times \mathbb{R} \) into \( X \) by \( \pi(x_\alpha t) = (x_\alpha t), \alpha \in I, \) then \( (X, \pi) \) is a dynamical system. The dynamical system \( (X, \pi) \), obtained above, is called the direct product or the product of the family \( (X_\alpha, \pi_\alpha), \alpha \in I \).

We take the usual definitions of positive limit set \( \Omega_x \), positive distal, positive Poisson stable, and positive Lagrange stable motions. As usual, we drop the word positive and we use the notations of [1, 4].

3. Main results

**Proposition 3.1.** Let \( (X_\alpha, \pi_\alpha), \alpha \in I, \) be a family of \{Lagrange stable\ \} \{distal\} s.d.s.
and \((X, \pi)\) the product s.d.s. Let \(x \in X\) and \(x = \{x_\alpha\}\), then \((X, \pi)\) is \{Lagrange stable\} {distal}.

**Proposition 3.2.** If a Lagrange stable motion is Poisson stable and distal, then \(\text{ClY}(x) = \Omega(x) = \Omega_x\).

**Proof.** The proof follows from [4, Thm. 2.1].

**Theorem 3.3.** Let \((X_\alpha, \pi_\alpha), \alpha \in I\), be a family of dynamical systems and \((X, \pi)\) the product of the dynamical systems. Let \(x \in X\) and \(x = \{x_\alpha\}\). Then \(\Omega_x(x) \subseteq \prod \Omega_{x_\alpha}(x_\alpha)\), where \(\Omega_{x_\alpha}(x_\alpha)\) is the positive limit set of \(x_\alpha\) in the dynamical systems \((X_\alpha, \pi_\alpha)\). (The two \(\pi\)'s have distinct meanings according to the context.)

Since, in general, the equality does not hold and \(\Omega_x\) may be empty, the Poisson stability in the constituent dynamical system may be lost from the product of the dynamical systems. Here, we find the conditions under which \(\Omega_x(x) = \prod \Omega_{x_\alpha}(x_\alpha), \alpha \in I\) and thus, the product of Poisson stable motions remains Poisson stable in the product system.

**Theorem 3.4.** If a compact motion is Poisson stable and distal, then it is a compact recurrent motion.

**Proof.** Let the motion \(\pi(x, t)\) be Poisson stable and distal, then its trajectory \(\text{Y}(x)\) is closed. Therefore,

\[\text{Y}(x) = \text{ClY}(x) = \Omega_x.\]  

(3.1)

As the motion is compact, each of the above sets is compact and minimal and thus, by Birkhoff recurrence theorem, \(\pi(x, t)\) is compact and recurrent.

**Theorem 3.5.** Let \((X, \pi)\) be a semi-dynamical system. Let \(\pi\) be a Lagrange stable, then \(\pi\) is distal if and only if, for every net \(t_i\) in \(\mathbb{R}^+\), the phase space

\[X = \{z \in X : xt_j \rightarrow z \text{ for some } x \in X \text{ and some subnet } t_j \text{ of } t_i\}\]  

(3.2)

\[2, \text{Thm. 2.6}].

**Theorem 3.6.** Let \((X, \pi)\) be Lagrange stable and distal s.d.s. then every net in the trajectory \(Y(x)\) of the Poisson stable motion \(\pi(x, t)\) is a Cauchy net.

**Proof.** Let \(Y(x)\) be the trajectory of the Poisson stable motion \(\pi(x, t)\) in s.d.s. \((X, \pi)\) which is Lagrange stable and distal. Let \(xt_n\) be a net in \(Y(x)\) which is compact (Proposition 3.2). Therefore, \(xt_n\) has a subnet, say \(xt_m\) with \(xt_m \rightarrow z\), i.e., \(z\) is a cluster point of \(xt_n\). Hence, \(xt_n\) is a Cauchy net.

**Theorem 3.7.** Let \((X_\alpha, \pi_\alpha), \alpha \in I\), be a family of Lagrange stable and distal s.d.s. and \((X, \pi)\) be the product s.d.s. Let \(x \in X\) and \(x = \{x_\alpha\}\). A motion \(\pi(x, t)\) is Poisson stable in \((X, \pi)\) if and only if \(\pi_\alpha(x_\alpha, t)\) is Poisson stable in \((X_\alpha, \pi_\alpha)\) for each \(\alpha \in I\).

**Proof.** Let \((x_\alpha, \pi_\alpha), \alpha \in I\), be a Lagrange stable and distal s.d.s. Let \(\pi(x_\alpha, t) = x_\alpha t\) be a Poisson stable motion in \((X_\alpha, \pi_\alpha), \alpha \in I\), then its trajectory \(Y_\alpha(x_\alpha)\) is compact and the net \(x_\alpha t_n, \alpha \in I\), is a Cauchy net in \(Y_\alpha(x_\alpha)\) (Theorem 3.6). Now, the Cauchy
nets $x_\alpha t_n$, $\alpha \in I$ yield the Cauchy net $xt_n$ in $Y(x)$ in $(X, \pi)$ [3, p. 194]. As the product of compact sets is a compact set, $Y(x)$ is compact and $xt_n$ is a net in compact $Y(x)$. Thus, it has a subnet $xt_m \to z$, i.e., $z$ is a cluster point of $xt_n$. Hence, $xt_n$ is frequently in every neighborhood $U$ of $z$. Given a neighborhood $U$ of $z$ for every $i \in A$, there is a $j \in A$, $i \geq J$ such that $xt_i \in U$ however $t_i \to +\infty$. Hence, $\pi(x, t)$ is Poisson stable. The converse follows from [3, Thm. 25, p. 194] which states that a net in the product is a Cauchy net if and only if its projection into each coordinate space is a Cauchy net.

**Theorem 3.8.** Let $(X_\alpha, \pi_\alpha)$, $\alpha \in I$, be a family of Lagrange stable distal s.d.s. Let $x \in X$, $x = \{x_\alpha\}$, and $(X, \pi)$ the product s.d.s. Let $Y_\alpha(x_\alpha)$, $\alpha \in I$, be the product of trajectories. Then $\Pi Y_\alpha(x_\alpha) = Y(x)$. Moreover,

$$\Pi \Omega_\alpha(x_\alpha) = \Omega_\chi(x).$$

(3.3)

**Proof.** Since each $Y_\alpha(x_\alpha)$, $\alpha \in I$, is closed and compact,

$$\text{Cl} \Pi Y_\alpha(x_\alpha) = \Pi \text{Cl} Y_\alpha(x_\alpha) = \text{Cl} Y(x),$$

(3.4)

$$\Pi Y_\alpha(x_\alpha) = Y(x).$$

(3.5)

Moreover,

$$\Pi \Omega_\alpha(x_\alpha) = \Omega_\chi(x).$$

(3.6)

References


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