Nonlinear Functional Integro-Differential Equations in Hilbert Space

J. Y. Park, S. Y. Lee, and M. J. Lee

(Received 7 November 1996 and in revised form 3 November 1997)

Abstract. Let $X$ be a Hilbert space and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$. We establish the existence and norm estimation of solutions for the parabolic partial functional integro-differential equation in $X$ by using the fundamental solution.

Keywords and phrases. Functional integro-differential equation, elliptic differential operators, fundamental solution, Gårding’s inequality, successive approximation, norm estimation.

1991 Mathematics Subject Classification. 35A05, 35J60, 45K05.

1. Introduction. Let $X$ be a Hilbert space and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$. We consider the following parabolic partial functional integro-differential equation.

\begin{equation}
\frac{\partial u}{\partial t} = A_0 u(t, x) + A_1 u(t - h, x) + \int_{-h}^{0} a(s) A_2 u(t + s, x) \, ds \\
+ \int_{0}^{t} \left\{ k(t, s) G(s, u(s - h), x) + H(t, s, u(s - h, x)) \right\} \, ds \\
+ F(t, u(t - h, x)) + f(t, x), \quad 0 < t \leq T, \quad x \in \Omega,
\end{equation}

where $A_i (i = 0, 1, 2)$ are elliptic differential operators, $f$ is a forcing function, $h > 0$ is a delay time, $a(s)$ is a real scalar function on $[-h, 0]$, $G, H$, and $F$ are nonlinear functions, and $k$ is a kernel. The boundary condition attached to (1.1) is, e.g., given by the Dirichlet boundary condition

\begin{equation}
u|_{\partial \Omega} = 0, \quad 0 < t \leq T,
\end{equation}

and the initial condition is given by

\begin{equation}u(\theta, x) = g(\theta, x), \quad \theta \in [-h, 0], \quad x \in \Omega.
\end{equation}

From [4], the above mixed problems (1.1), (1.2), and (1.3) can be formulated abstractly as

\begin{equation}
\frac{du(t)}{dt} = A_0 u(t) + A_1 u(t - h) + \int_{-h}^{0} a(s) A_2 u(t + s) \, ds \\
+ \int_{0}^{t} \left\{ k(t, s) G(s, u_{s}) + H(t, s, u_{s}) \right\} \, ds \\
+ F(t, u_{t}) + f(t), \quad 0 < t \leq T, \\
u(\theta) = g(\theta), \quad \theta \in [-h, 0],
\end{equation}
where the state $u(x)$ of the system (1.5) lies in an appropriate Hilbert space and $A_i(i = 0, 1, 2)$ are unbounded operators associated with $\mathcal{A}_i(i = 0, 1, 2)$, respectively. Next, we explain the notation $u_t$ in (1.5). Let $I = [-h, 0]$. If a function $u(t)$ is continuous from $I \cup [0, T]$ into a Hilbert space $X$, then $u_t$ is an element in $C = C([-h, 0]; X)$, which has the point-wise definition

$$u_t(\theta) = u(t + \theta) \quad \text{for} \ \theta \in I. \quad (1.6)$$

Let $\Delta_T = \{(s, t); 0 \leq s \leq t \leq T\}$. We assume in (1.5) that $G : [0, T] \times C \to X$, $H : \Delta_T \times C \to X$, $F : [0, T] \times C \to X$ and the kernel $k : \Delta_T \to R$ ($R$ denotes the set of real numbers) are continuous, $f : [0, T] \to V^*$ with some enlarged space $V^* \supset H$ and $g : [-h, 0] \to V$ with some dense subspace $V \subset H$. It is assumed that the inclusions $V \subset H \subset V^*$ are continuous and $V^*$ is the dual space of $V$.

Many authors [2, 8] studied the following delay differential equation:

$$\frac{du(t)}{dt} = A_0 u(t) + A_1 u(t-h) + \int_{-h}^{0} a(s) A_2 u(t+s) \, ds + f(t), \quad \text{a.e. } t \geq 0, \quad (2.1)$$

$$u(0) = g^0, \quad u(s) = g^1(s), \quad \text{a.e. } s \in [-h, 0].$$

Let $a(u, v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding’s inequality

$$\text{Re} \ a(u, u) \geq c_0 \|u\|^2 - c_1 |u|^2, \quad (2.2)$$

where $c_0 > 0$ and $c_1 \geq 0$ are real constants. Let $A_0$ be the operator associated with this sesquilinear form

$$\langle v, A_0 u \rangle = -a(u, v), \quad u, v \in V, \quad (2.3)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V$ and $V^*$. The operator $A_0$ is bounded linear from $V$ into $V^*$. The realization of $A_0$ in $H$, which is the restriction of $A_0$ to the domain $D(A_0) = \{ u \in V : A_0 u \in H \}$, is also denoted by $A_0$. It is proved in Tanabe [6] that $A_0$ generates an analytic semigroup $e^{tA_0} = T(t)$ both in $H$ and $V^*$ and that $T(t) : V^* \to V$ for each $t > 0$. Throughout this paper, it is assumed that each $A_i(i = 1, 2)$ is bounded and linear from $V$ to $V^*$ (i.e., $A_i \in \mathcal{L}(V, V^*)$) such that $A_i$ maps $D(A_0)$
endowed with the graph norm of $A_0$ to $H$ continuously. The real valued scalar function $a(s)$ is assumed to be Hölder continuous on $[-h,0]$. We introduce a Stieltjes measure $\eta$ given by
\[
\eta(s) = -\chi_{[-\infty,-h]}(s)A_1 - \int_s^0 a(\xi)\,d\xi\,A_2 : V \to V^* , \quad s \in [-h,0],
\]
where $\chi_{[-\infty,-h]}$ denotes the characteristic function of $(-\infty,-h]$. Then the delay term in (2.1) is written simply as $\int_{-h}^0 d\eta(s)u(t+s)$. The fundamental solution $W(t)$ of (2.1) is defined as a unique solution of
\[
W(t) = \begin{cases} T(t) + \int_0^t T(t-s)\int_{-h}^0 d\eta(s)W(s)\,ds, & t \geq 0, \\ 0, & t < 0, \end{cases}
\]
and $W(t)$ is constructed by Tanabe [7] under the Hölder continuity of $a(s)$.

**Theorem 2.1** [2]. The fundamental solution $W(t)$ is strongly continuous in $V,H$, and $V^*$, and for each $t > 0$, $W(t) : V^* \to V$. Furthermore, $W(t)$ satisfies
\[
\frac{d}{dt} W(t) = A_0 W(t) + \int_{-h}^0 d\eta(s)W(t+s), \quad \text{a.e. } t > 0.
\]

For each $t > 0$, we define the operator valued function $U_t(\cdot)$ by
\[
U_t(s) = \int_{-h}^s W(t-s+\xi)\,d\eta(\xi) : V \to V, \quad \text{a.e. } s \in [-h,0].
\]

Let $T > 0$ be fixed. Associated with $U_t(\cdot)$, we consider the operator $\mathcal{U} : L^2(0,-h;V) \to L^2(0,T;V)$ defined by
\[
(\mathcal{U}g^1)(t) = \int_{-h}^0 U_t(s)g^1(s)\,ds, \quad t \in [0,T]
\]
for $g^1 \in L^2(0,-h;V)$.

**Theorem 2.2** [8]. Let $T > 0$ be fixed. Assume that $f \in L^2(0,T;V^*)$ and $g = (g^0,g^1) \in H \times L^2(0,-h;V)$. Then there exists a unique solution $u(t) = u(t;f,g)$ of (2.1) on $[0,T]$ satisfying
\[
u \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).
\]

Further, for each $T > 0$, there is a constant $K_T$ such that
\[
\int_0^T \|u(t)\|^2\,dt + \int_0^T \|\frac{du(t)}{dt}\|^2\,dt \leq K_T \left( \|g^0\|^2 + \int_{-h}^0 \|g^1(s)\|^2\,ds + \int_0^T \|f(t)\|^2\,dt \right).
\]

This solution $u(t)$ is represented by
\[
u(t;f,g) = W(t)g^0 + (\mathcal{U}g^1)(t) + \int_0^t W(t-s)f(s)\,ds.
\]

In what follows, in order to consider the solutions in the state space $C = C([-h,0];H)$, we assume that $g = (g^0,g^1)$ is continuous in $H$, i.e.,
\[
g(0) = g^0, \quad g(\cdot) = g^1(\cdot) \in C([-h,0];H).
\]
Let
\[ \hat{u}(t;f,g) = \begin{cases} u(t;f,g), & t \in [0,T], \\ g(t), & t \in [-h,0]. \end{cases} \] (2.13)

Then, by Theorem 2.2, we get
\[ \hat{u}(\cdot;f,g) \in C([-h,T];H) \] (2.14)
if (2.12) is satisfied.

3. Existence and uniqueness of functional integro-differential equations. Using the fundamental solution \( W(t) \) in Section 2, we consider the following abstract functional integral equation.
\[ v(t) = u(t;f,g) + \int_0^t W(t-s) \left[ \int_0^s \{ k(s,\tau) G(\tau,v_\tau) + H(s,\tau,v_\tau) \} d\tau + F(s,v_s) \right] ds, \quad 0 < t \leq T, \]
\[ v(\theta) = g(\theta), \quad \theta \in [-h,0], \] (3.1)

where \( u(t;f,g) \) is given by (2.11).

We list the following hypotheses.

(A1) The nonlinear functions \( G : [0,T] \times C \rightarrow H, H : \Delta_T \times C \rightarrow H, F : [0,T] \times C \rightarrow H, \)
and the kernel \( k : \Delta_T \rightarrow \mathbb{R} \) (\( \mathbb{R} \) denotes the set of real numbers) are continuous.

(A2) Let \( b_1, b_2 \) be continuous functions such that
\[ |G(t,\phi) - G(t,\overline{\phi})|_X \leq b_1(t) |\phi - \overline{\phi}|_C; \]
\[ |H(t,s,\phi) - H(t,s,\overline{\phi})|_X \leq b_2(t,s) |\phi - \overline{\phi}|_C; \]
\[ |F(t,\phi) - F(t,\overline{\phi})|_X \leq b_3(t) |\phi - \overline{\phi}|_C \] (3.2)
for \( t,s \in [0,T], \phi, \overline{\phi} \in C \).

(A3) The function \( k(t,s) \) is Hölder continuous with exponent \( \alpha \), i.e., there exists a positive constant \( a \) such that
\[ |k(t_1,s_1) - k(t_2,s_2)| \leq a (|t_1 - t_2|^\alpha + |s_1 - s_2|^\alpha) \] (3.3)
for \( t_1, t_2, s_1, s_2 \in [0,T], \ 0 < \alpha \leq 1 \).

(A4) For all \( 0 \leq s \leq t \leq T, \)
\[ G(t,0) = 0, \quad H(t,s,0) = 0, \quad F(t,0) = 0. \] (3.4)

**Theorem 3.1.** Let \( f \in L^2(0,T;V^*) \) and \( g = (g(0),g(\cdot)) \in H \times L^2(-h,0;V) \) satisfy (2.12). Assume that the hypotheses (A1)–(A4) hold. Then there exists a time \( t_1 > 0 \) such that the functional integral equation (3.1) admits a unique solution \( v(t) \) on \([0,t_1]\).

**Proof.** We prove this theorem by using the method of successive approximations.

Set \( v^0(t) = u(t;f,g), \ t \geq 0 \). Let \( \hat{v}^0(t) \) be the extension of \( v^0(t) \) on \([-h,T]\) by (2.13). Then, by the assumptions on \( f \) and \( g \), we have \( \hat{v}^0(t) \in C([-h,T];H) \). By hypotheses (A1)–(A4), we define \( \{ \hat{v}^n \}_{n=0}^\infty \subset C([-h,T];H) \) successively by
\( \hat{v}^n(t) = u(t; f, g) \)
\[
+ \int_0^t W(t-s) \left[ \int_0^s \{ k(s, \tau) G(\tau, \hat{v}^n_\tau) + H(s, \tau, \hat{v}^{n-1}_\tau) \} d\tau + F(s, \hat{v}^n_\tau) \right] ds,
\]
\[ 0 < t \leq T, \]
\[ \hat{v}^n(\theta) = g(\theta), \quad \theta \in [-h, 0]. \]

It is obvious that \( M = \sup_{t \in [0, T]} |W(t)|_{L(H)} \) is finite and that
\[ \hat{v}^{n+1}(\theta) - \hat{v}^n(\theta) = 0, \quad \theta \in [-h, 0]. \]

For \( 0 \leq t \leq T \), we have, by (A1)-(A4) and the strong continuity of \( W(t) \) on \([0, T]\),
\[
| \hat{v}^{n+1}(t) - \hat{v}^n(t) |
= \left| \int_0^t W(t-s) \left[ \int_0^s \{ k(s, \tau) G(\tau, \hat{v}^n_\tau) + H(s, \tau, \hat{v}^n_\tau) \} d\tau + F(s, \hat{v}^n_\tau) \right] ds 
\right|
\[ - \int_0^t W(t-s) \left[ \int_0^s \{ k(s, \tau) G(\tau, \hat{v}^{n-1}_\tau) + H(s, \tau, \hat{v}^{n-1}_\tau) \} d\tau + F(s, \hat{v}^{n-1}_\tau) \right] ds \]
\[ \leq M \int_0^t \left[ \int_0^s |k(s, \tau)| |G(\tau, \hat{v}^n_\tau) - G(\tau, \hat{v}^{n-1}_\tau)| d\tau ds \right] 
+ \int_0^t |H(s, \tau, \hat{v}^n_\tau) - H(s, \tau, \hat{v}^{n-1}_\tau)| d\tau \] 
\[ + \int_0^t |F(s, \hat{v}^n_\tau) - F(s, \hat{v}^{n-1}_\tau)| ds \]
\[ \leq M \int_0^t \left[ \int_0^s \left\{ |k(s, \tau)| |b_1(\tau)| ||\hat{v}^n_\tau - \hat{v}^{n-1}_\tau|| + |b_2(s, \tau)| ||\hat{v}^n_\tau - \hat{v}^{n-1}_\tau|| \right\} d\tau \right] ds 
+ M \int_0^t |b_3(s)| ||\hat{v}^n_s - \hat{v}^{n-1}_s|| ds 
\leq M \int_0^t [KL_1 + L_2] ||\hat{v}^n_\tau - \hat{v}^{n-1}_\tau|| ds 
+ M \int_0^t L_3 ||\hat{v}^n_s - \hat{v}^{n-1}_s|| ds 
\leq [M(KL_1 + L_2) \frac{1}{2} t^2 + ML_3] ||\hat{v}^n - \hat{v}^{n-1}||_{C([-h,T]; H)} 
= (c_1 t + c_2) t ||\hat{v}^n - \hat{v}^{n-1}||_{C([-h,T]; H)}, \]

where \( c_1 = (1/2)M(KL_1 + L_2) \) and \( c_2 = ML_3 \). We now choose a sufficiently small constant \( t_1 > 0 \) such that
\[ L = (c_1 t_1 + c_2) t_1 < 1. \]

Then by (3.6), (3.8), and (3.9), we get
\[
||\hat{v}^{n+1} - \hat{v}^n||_{C([-h,T]; H)} \leq L ||\hat{v}^n - \hat{v}^{n-1}||_{C([-h,T]; H)} 
\leq \cdots \cdots \cdots \leq \cdots \cdots \cdots 
\leq L^n ||\hat{v}^1 - \hat{v}^0||_{C([-h,T]; H)}. \]

This implies that \( \{ \hat{v}^n \}_{n=0}^\infty \) converges uniformly to some \( \hat{v} \in C([-h,0]; H) \). Therefore,
\[ \lim_{n \to \infty} \sup_{t \in [0,t_1]} ||\hat{v}^n_t - \hat{v}_t||_{C([-h,0]; H)} = 0. \]

Hence, by letting \( n \to \infty \) in (3.5), in view of (A1)–(A4) and (3.11), we get

\[
\hat{v}(t) = u(t; f, g) + \int_0^t \left[ W(t-s) \left[ \int_0^s \{ k(s, \tau) G(\tau, \hat{v}_\tau) \ight. + H(s, \tau, \hat{v}_\tau) \} d\tau \right] ds, \quad 0 < t \leq t_1,
\]
\[
\hat{v}(\theta) = g(\theta), \quad \theta \in [-h, 0].
\]

This shows the local existence of a solution \( v(t) = \hat{v}(t)|_{[0,t_1]} \) of (3.1) on \([0,t_1]\). Let \( v_1 \) and \( v_2 \) be the solution of (3.1) on \([0,t_1]\). Then it is easy to see, similarly to the above, that

\[
||\hat{v}^1 - \hat{v}^2||_{C([-h,t_1]; H)} \leq L ||\hat{v}^1 - \hat{v}^2||_{C([-h,t_1]; H)},
\]

so that by \( L < 1 \), \( v^1(t) = v^2(t) \) on \([0,t_1]\). This proves the uniqueness.

Since \( k(s, \tau) G(\tau, \hat{v}_\tau), H(s, \tau, \hat{v}_\tau), F(s, v_s) \in L^2(0,t_1; H) \subset L^2(0,t_1; V^*) \), by Theorem 2.1, we see that the solution \( v(t) \) of (3.1) satisfies

\[
\frac{dv(t)}{dt} = A_0 v(t) + A_1 v(t-h) + \int_{-h}^t a(s) A_2 v(t+s) ds
\]
\[
+ \int_0^t \left[ k(t,s) G(s,v_s) + H(t,s,v_s) \right] ds + F(t,v_t) + f(t), \quad \text{a.e. } t \in [0,t_1],
\]
\[
v(\theta) = g(\theta), \quad \theta \in [-h,0],
\]

and \( v \in L^2(0,t_1; V) \cap W^{1,2}(0,t_1; V^*) \). In this sense, we call this \( v \) a mild solution of (1.5) on \([0,t_1]\). We give a norm estimation of the mild solution of (1.5) and establish the global existence of solutions with the aid of norm estimations. It is well known (cf. Lions and Magenes [3, Prop. 2.1, Thm. 3.1]) that the inclusion \( L^2(0,T; V) \cap W^{1,2}(0,T; V^*) \subset C([0,T]; H) \) is continuous, that is, there exists a constant \( c_0 \) such that

\[
\|u\|_{C([0,T];H)} \leq c_0 \left( \|u\|_{L^2(0,T;V)} + \left\| \frac{du}{dt} \right\|_{L^2(0,T;V^*)} \right),
\]

for all \( u \in L^2(0,T; V) \cap W^{1,2}(0,T; V^*) \).

**Lemma 3.1** [5]. Let \( a(t), b(t), \) and \( c(t) \) be real valued nonnegative continuous functions defined on \( R^+ \), for which the inequality

\[
c(t) \leq c_0 + \int_0^t a(s) c(s) ds + \int_0^t a(s) \left[ \int_0^s b(\tau) c(\tau) d\tau \right] ds
\]

holds for all \( t \in R^+ \), where \( c_0 \) is a nonnegative constant. Then

\[
c(t) \leq c_0 \left( 1 + \int_0^t a(s) \exp \left[ \int_0^s (a(\tau) + b(\tau)) d\tau \right] ds \right) \quad \text{for all } t \in R^+.
\]
By using Lemma 3.1, we get
\[ \|v(t)\|_{C([-h,T];\mathbb{R})} \leq c \left( \|g(0)\| + \|g\|_{L^2(-h,0;\mathbb{R})} + \|f\|_{L^2(0,T;\mathbb{R}^n)} \right) e^{Kt}, \] (3.18)
where \( c \) is a positive constant which does not depend on \( v \).

**Proof.** From hypotheses \((A_1)-(A_4)\), we have
\[
|v(t + \theta;f,g)| \leq |u(t + \theta;f,g)| + \left| \int_0^{t+\theta} W(t + \theta - s) \right. \\
\times \left[ \int_0^s \left[ k(s,\tau)G(\tau,v_\tau) + H(s,\tau,v_\tau) \right] d\tau + F(s,v_s) \right] ds \\
\leq \|u(\cdot;f,g)\|_{C([-h,T];\mathbb{R})} \\
+ M \left[ \int_0^{t+\theta} \right. \\
\left. \left[ \int_0^s \left[ K|\mathbf{b}(1,\tau)| \|v_\tau\| + |\mathbf{b}_2(s,\tau)| \|v_\tau\| \right] d\tau + |\mathbf{b}_3(s)| \|v_s\| \right] ds. \] (3.19)

Hence, by (2.10) and (3.15),
\[
\|v(t + \theta;f,g)\| = \sup_{\theta \in [-h,0]} |v(t + \theta;f,g)| \\
\leq K_T c_0 \left( \|g(0)\| + \|g\|_{L^2(-h,0;\mathbb{R})} + \|f\|_{L^2(0,T;\mathbb{R}^n)} \right) \\
+ \int_0^t c_1 \|v_s(\cdot;f,g)\| ds + \int_0^t \int_0^s c_2 \|v_\tau(\cdot;f,g)\| d\tau ds \\
\leq c' \left( \|g(0)\| + \|g\|_{L^2(-h,0;\mathbb{R})} + \|f\|_{L^2(0,T;\mathbb{R}^n)} \right) \\
+ M \left( \int_0^t \|v_s(\cdot;f,g)\| ds + \int_0^t \int_0^s \|v_\tau(\cdot;f,g)\| d\tau ds \right). \] (3.20)

By using Lemma 3.1, we get
\[
\|v(t;f,g)\|_{C([-h,T];\mathbb{R})} \leq c \left( \|g(0)\| + \|g\|_{L^2(-h,0;\mathbb{R})} + \|f\|_{L^2(0,T;\mathbb{R}^n)} \right) \\
\times \left( 1 + M \int_0^t \exp \left( \int_0^s (M+1) d\tau \right) ds \right) \\
\leq c' \left( \|g(0)\| + \|g\|_{L^2(-h,0;\mathbb{R})} + \|f\|_{L^2(0,T;\mathbb{R}^n)} \right) \\
\times [1 + M \exp \{(M+1)t\} t] \\
\leq c \left( \|g(0)\| + \|g\|_{L^2(-h,0;\mathbb{R})} + \|f\|_{L^2(0,T;\mathbb{R}^n)} \right) e^{Kt} \] (3.21)
for some constants \( c \) and \( K \). This completes the proof. \( \square \)

**Theorem 3.3.** Assume that the conditions in Theorem 3.1 hold. Then there exists a unique solution \( v(t) \) on \([0,T]\) of (3.1) which satisfies the estimate
\[
\|v(\cdot;f,g)\|_{C([0,T];\mathbb{R})} \leq c \left( \|g(0)\| + \|g\|_{L^2(-h,0;\mathbb{R})} + \|f\|_{L^2(0,T;\mathbb{R}^n)} \right) e^{KT} \] (3.22)
for some constants \( c \) and \( K \).
ACKNOWLEDGEMENT. This work was supported by KOSEF, 1996.

REFERENCES


PARK, Y. LEE, AND J. LEE: DEPARTMENT OF MATHEMATICS, PUSAN NATIONAL UNIVERSITY, PUSAN 609-735, KOREA