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FIXED POINT THEOREMS FOR GENERALIZED LIPSCHITZIAN SEMIGROUPS IN BANACH SPACES

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Abstract. Fixed point theorems for generalized Lipschitzian semigroups are proved in p-uniformly convex Banach spaces and in uniformly convex Banach spaces. As applications, its corollaries are given in a Hilbert space, in L^p spaces, in Hardy space H^p, and in Sobolev spaces $H^{k,p}$, for $1 < p < \infty$ and $k \geq 0$.

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1. Introduction. Let K be a nonempty closed convex subset of a Banach space E. A mapping $T : K \longrightarrow K$ is said to be Lipschitzian mapping if for each $n \geq 1$, there exists a positive real number k_n such that

$$||T^n x - T^n y|| \leq k_n ||x - y|| \tag{1}$$

for all x, y in K. A Lipschitzian mapping is said to be nonexpansive if $k_n = 1$ for all $n \geq 1$, uniformly k-Lipschitzian if $k_n = k$ for all $n \geq 1$, and asymptotically nonexpansive if $\lim_n k_n = 1$, respectively. These mappings were first studied by Geobel and Kirk [6] and Geobel, Kirk, and Thele [8]. Lifshitz [10] showed that in a Hilbert space H, a uniformly k-Lipschitzian mapping T with $k < \sqrt{2}$ has a fixed point. Downing and Ray [3] and Ishihara and Takahashi [9] verified that Lifshitz's theorem is valid for uniformly Lipschitzian semigroup in Hilbert spaces.

Mizoguchi and Takahashi [14] introduced the notion of a submean on an appropriate space and, using a submean, they proved a fixed point theorem for uniformly Lipschitzian semigroup in a Hilbert space. Recently, Tan and Xu [21] generalized the result of Mizoguchi and Takahashi [14] to a Banach space setting and, also, proved a new fixed point theorem for uniformly k-Lipschitzian semigroup in a uniformly convex Banach space.

Now, we consider the following class of mappings, which we call generalized Lipschitzian mapping whose nth iterate T^n satisfies the following condition:

$$\begin{aligned} ||T^n x - T^n y|| \leq a_n ||x - y|| &+ b_n (||x - T^n x|| + ||y - T^n y||) \\ &+ c_n (||x - T^n y|| + ||y - T^n x||) \end{aligned} \tag{2}$$

for each $x, y \in K$ and $n \geq 1$, where a_n, b_n, c_n are the nonnegative constants such that there exists an integer n_0 such that $b_n + c_n < 1$ for all $n \geq n_0$.

This class of generalized Lipschitzian mappings are more general than nonexpansive, asymptotically nonexpansive, Lipschitzian, and uniformly k-Lipschitzian mappings and it can be seen by taking $b_n = c_n = 0$.

In this paper, we prove some fixed point theorems for generalized Lipschitzian semigroups in p-uniformly convex Banach spaces and in uniformly convex Banach spaces. Next, we give its corollaries in a Hilbert space, in L^p spaces, in Hardy space H^p, and in Sobolev spaces $H^{k,p}$, for $1 < p < \infty$ and $k \ge 0$. Our results improve and extend results from [9, 14, 21, 22].

2. Preliminaries. Let $p > 1$ and denote by λ the number in $[0,1]$ and by $w_p(\lambda)$ the function $\lambda \cdot (1-\lambda)^p + \lambda^p \cdot (1-\lambda)$. The functional $\|\cdot\|^p$ is said to be uniformly convex (cf. Zalinescu [24]) on the Banach space E if there exists a positive constant c_p such that, for all $\lambda \in [0,1]$ and $x, y \in E$, the following inequality holds:

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda \|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda) \cdot c_p \cdot \|x-y\|^p. \tag{3}$$

Xu [23] proved that the functional $\|\cdot\|^p$ is uniformly convex on the whole Banach space E if and only if E is p-uniformly convex, i.e., there exists a constant $c_p > 0$ such that the modulus of convexity (see [7]) $\delta_E(\epsilon) \ge c_p \cdot \epsilon^p$ all $0 \le \epsilon \le 2$.

Let G be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that, for each $a \in G$, the mapping $t \longrightarrow a \cdot t$ and $t \longrightarrow t \cdot a$ from G onto itself are continuous. A semitopological semigroup G is left reversible if any two closed right ideals of G have nonempty intersection. In this case, $(G, \preceq)$ is a directed system when the binary relation "$\preceq$" on G is defined by $a \preceq b$ if and only if $\{a\} \cup \overline{aG} \supseteq \{b\} \cup \overline{bG}$, where $\overline{D}$ is the closure of set D. Examples of left reversible semigroups include commutative and all left amenable semigroups.

Let $m(G)$ be the Banach space of bounded real valued functions on G with the supremum norm. Suppose X is a subspace of $m(G)$ containing constants. Following Mizoguchi and Takahashi [14], we say that a real valued function μ on X is a submean on X if the following conditions are satisfied:

(i) $\mu(f+g) \le \mu(f) + \mu(g)$ for all $f, g \in X$;
(ii) $\mu(\alpha f) = \alpha\mu(f)$ for all $f \in X$ and $\alpha \ge 0$;
(iii) if $f, g \in X$ with $f \le g$, then $\mu(f) \le \mu(g)$; and
(iv) $\mu(c) = c$ for every constant c.

If μ is a submean on X and $f \in X$, then we denote by either $\mu(f)$ or $\mu_t(f(t))$, according to time and circumstances, the value of μ at f. For $a \in G$ and $f \in m(G)$, we define $(l_a f)(t) = f(at)$ and $(r_a f)(t) = f(ta)$ for all $t \in G$. Let X be a subspace of $m(G)$ containing constants which is l_G-invariant, i.e., $l_a(X) \subseteq X$ for all $a \in G$. Then a submean μ on x is said to be left invariant if $\mu(f) = \mu(l_a f)$ for every $a \in G$ and $f \in X$. A right invariant submean is defined similarly. A submean is called invariant if it is left and right invariant. Let K be a closed convex subset of a Banach space E. Then a collection $\mathcal{S} = \{T_s : s \in G\}$ of mappings of K into itself is said to be a generalized Lipschitzian semigroup on K if the following conditions are satisfied:

(i) $T_{st}x = T_s T_t x$ for all $s, t \in G$ and $x \in K$;
(ii) for each $x \in K$, the mapping $t \longrightarrow T_t x$ from G into K is continuous; and

(iii) for each $s \in G$

$$\|T_s x - T_s y\| \leq a_s \|x - y\| + b_s (\|x - T_s x\| + \|y - T_s y\|) + c_s (\|x - T_s y\| + \|y - T_s x\|), \tag{4}$$

for $x, y \in K$, where $a_s, b_s, c_s > 0$ such that there exists a $t_1 \in G$ such that $b_s + c_s < 1$ for all $s \succeq t_1$.

The following lemma is needed to prove the main result:

Lemma 1 [22, Lem. 2.1]. *Let E be a p-uniformly convex Banach space, K a nonempty closed convex subset of E, and $\{x_t : t \in G\}$ a bounded family of elements of E. Also, suppose that for every x in K, the function f on G, defined by*

$$f(t) = \|x_t - x\|^p, \quad t \in G, \tag{5}$$

belongs to X. Set

$$r(x) = \mu_t \|x_t - x\|^p, \quad x \in K \tag{6}$$

and

$$r = \inf\{r(x) : x \in K\}. \tag{7}$$

Then there exists a unique point z in K such that

$$r + c_p \|z - x\|^p \leq r(x) \tag{8}$$

for all x in K, where c_p is the constant appearing in (3).

3. Main results.

Now, we prove the first result of this paper.

Theorem 1. *Let K be a nonempty closed convex subset of a p-uniformly convex Banach space E, X an l_G-invariant subspace of $m(G)$ containing constants which has left invariant submean μ, and $\mathcal{S} = \{T_s : s \in G\}$ a generalized Lipschitzian semigroup on K. Suppose that there exists an x_0 in K such that $\{T_s x_0 : x \in G\}$ is bounded and that, for every $u, v \in K$, the function f on G defined by*

$$f(t) = \|T_t u - v\|^p, \quad t \in G, \tag{9}$$

and the function g on G defined by

$$g(t) = 2^{p-1}(\alpha_t^p + \beta_t^p), \quad t \in G \tag{10}$$

belong to X. Then, if $2^{p-1}\{\mu_t(\alpha_t^p + \beta_t^p)\} < 1 + c_p$, where $\alpha_t = (a_t + b_t + c_t)/(1 - b_t - c_t)$, $\beta_t = (2b_t + 2c_t)/(1 - b_t - c_t)$, and c_p is the constant appearing in (3), there exists a $z \in K$ such that $T_s z = z$ for all $s \in G$.

Proof. Since $\{T_s x_0 : s \in G\}$ is bounded, it follows that $\{T_s x : s \in G\}$ is bounded for every $x \in K$. By Lemma 1, we inductively construct a sequence $\{x_n\}_{n=1}^{\infty}$ in K in the following manner:

$$\mu_t \|T_t x_{n-1} - x_n\|^p = \min_{y \in K} \mu_t \|T_t x_{n-1} - y\|^p \tag{11}$$

for $n = 1, 2, \ldots$. It follows from Lemma 1 that

$$c_p \|x_n - y\|^p \le \mu_t \|T_t x_{n-1} - y\|^p - \mu_t \|T_t x_{n-1} - x_n\|^p \tag{12}$$

for all $y \in K$ and $n \ge 1$. Since T is generalized Lipschitzian, we get, after a simple calculation,

$$\|T_s x - T_s y\| \le \alpha_s \|x - y\| + \beta_s \|y - T_s y\| \tag{13}$$

for each $x, y \in K$ and $s \in G$, where $\alpha_s = (a_s + b_s + c_s)/(1 - b_s - c_s)$ and $\beta_s = (2b_s + 2c_s)/(1 - b_s - c_s)$. By putting $y = T_s x_n$ into (12), we have

$$\begin{aligned} c_p \|x_n - T_s x_n\|^p &\le \mu_t \|T_t x_{n-1} - T_s x_n\|^p - \mu_t \|T_t x_{n-1} - x_n\|^p \\ &= \mu_t \|T_{st} x_{n-1} - T_s x_n\|^p - \mu_t \|T_t x_{n-1} - x_n\|^p \\ &= \mu_t \|T_s T_t x_{n-1} - T_s x_n\|^p - \mu_t \|T_t x_{n-1} - x_n\|^p \\ &\le \mu_t \Big[\alpha_s \|T_t x_{n-1} - x_n\| + \beta_s \|x_n - T_s x_n\|\Big]^p - \mu_t \|T_t x_{n-1} - x_n\|^p \end{aligned} \tag{14}$$

or

$$(c_p - 2^{p-1}\beta_s^p)\|x_n - T_s x_n\|^p \le (2^{p-1}\alpha_s^p - 1) \cdot \mu_t \|T_t x_{n-1} - x_n\|^p. \tag{15}$$

Therefore, we have

$$\mu_s \|x_n - T_s x_n\|^p \le A \cdot \mu_t \|T_t x_{n-1} - x_n\|^p, \tag{16}$$

where $A = (2^{p-1}\alpha_s^p - 1)/(c_p - 2^{p-1}\beta_s^p) < 1$ by the assumption of the theorem. Since

$$\mu_t \|T_t x_{n-1} - x_n\|^p \le \mu_t \|T_t x_{n-1} - x_{n-1}\|^p \tag{17}$$

by (11), it follows from (13) that

$$\begin{aligned} \mu_t \|T_t x_{n-1} - x_n\|^p &\le A \cdot \mu_t \|T_t x_{n-1} - x_{n-1}\|^p \\ &\le A^n \mu_t \|T_t x_0 - x_0\|^p. \end{aligned} \tag{18}$$

Noticing that

$$\|x_n - x_{n-1}\|^p \le 2^{p-1}\Big(\|x_n - T_t x_{n-1}\|^p + \|T_t x_{n-1} - x_{n-1}\|^p\Big), \tag{19}$$

we get

$$\begin{aligned} \|x_n - x_{n-1}\|^p &\le 2^{p-1}\Big(\mu_t \|x_n - T_t x_{n-1}\|^p + \mu_t \|T_t x_{n-1} - x_{n-1}\|^p\Big) \\ &\le 2^p \mu_t \|T_t x_{n-1} - x_{n-1}\|^p \\ &\le 2^p A^{n-1} \mu_t \|T_t x_0 - x_0\|^p, \end{aligned} \tag{20}$$

which shows that $\{x_n\}$ is a Cauchy sequence and, hence, convergent. Let $z = \lim_{n\to\infty} x_n$. Then, for each $s \in G$, we have

$$\begin{aligned} \|z - T_s z\|^p &\le \Big(\|z - x_n\| + \|x_n - T_s x_n\| + \|T_s x_n - T_s z\|\Big)^p \\ &\le \Big[(1+\alpha_s)\|z - x_n\| + (1+\beta_s)\|x_n - T_s x_n\|\Big]^p \\ &\le 2^{p-1}\Big[(1+\alpha_s)^p\|z - x_n\| + (1+\beta_s)^p \cdot A \cdot \mu_t \|x_n - T_s x_n\|^p\Big] \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{21}$$

Therefore, $T_s z = z$ for all $s \in G$ and the proof is complete. □

Let E be a Banach space, K a nonempty closed convex subset of E, and G an unbounded subset of $[0,\infty)$ such that

$$t + h \in G \quad \text{for all } t, h \in G \tag{22}$$

and

$$t - h \in G \quad \text{for all } t, h \in G \quad \text{with } t > h \tag{23}$$

(e.g., $G = [0,\infty)$ or $G = N$, the set of nonnegative integers). Suppose $\mathcal{S} = \{T_s : s \in G\}$ is a generalized uniformly Lipschitzian semigroup on K, i.e., a family of self-mappings of K satisfying the conditions:

(i) $T_{s+h}x = T_s T_h x$ for all $s, h \in G$ and $x \in K$;

(ii) for each $x \in K$, the mappings $s \longrightarrow T_s x$ from G onto K is continuous when G has the relative topology of $[0,\infty)$; and

(iii)

$$\|T_s x - T_s y\| \le a\|x - y\| + b\Big(\|x - T_s x\| + \|y - T_s y\|\Big) + c\Big(\|x - T_s y\| + \|y - T_s x\|\Big) \tag{24}$$

for all x, y in K and s in G, where a, b, c are nonnegative constants such that $b + c < 1$.

For the rest of this paper, $\lim_t$ and $\overline{\lim}_t$ always stand for $\lim_{t\to\infty, t\in G}$, $\overline{\lim}_{t\to\infty, t\in G}$ respectively.

The normal structure coefficient $N(E)$ of E is defined (cf. [2]) by

$$N(E) = \inf\left\{\frac{\operatorname{diam} K}{r_K(K)} : K \text{ is a bounded convex subset of } E \text{ consisting of more than one point}\right\}, \tag{25}$$

where $\operatorname{diam} K = \sup\{\|x - y\| : x, y \in K\}$ is the diameter of K and $r_K(K) = \inf_{x\in K}\{\sup_{y\in K}\|x - y\|\}$ is the Chebyshev radius of K relative to itself. E is said to have uniformly normal structure if $N(E) > 1$. It is known that a uniformly convex Banach space has the uniformly normal structure and for a Hilbert space H, $N(H) = \sqrt{2}$. Recently, Pichugov [15] (cf. Prus [17]) showed that

$$N(L^p) = \min\{2^{1/p}, 2^{(p-1)/p}\}, \quad 1 < p < \infty. \tag{26}$$

Some estimate for normal structure coefficient in other Banach spaces may be found in [18].

Suppose E is a uniformly convex Banach space. Then it is easily seen that the equation

$$\xi^2 \delta_E^{-1}\left(1-\frac{1}{\xi}\right)\tilde{N}(E)=1 \tag{27}$$

has a unique solution $\xi > 1$, where $\tilde{N}(E) = N(E)^{-1}$.

Now, recall the definition of an asymptotic center. Let K be a nonempty closed convex subset of a Banach space E and $\{x_t : t \in G\}$ be a bounded family of elements of E. Then the asymptotic radius and asymptotic center of $\{x_t\}_{t\in G}$ with respect to K are the number

$$r_K(\{x_t\}) = \inf_{y\in K} \overline{\lim_t} \|x_t - y\| \tag{28}$$

and the (possibly empty) set

$$A_K(\{x_t\}) = \left\{y \in K : \overline{\lim_t} \|x_t - y\| = r_K(\{x_t\})\right\}, \tag{29}$$

respectively. It is easy to see that if E is reflexive, then $A_K(\{x_t\})$ is nonempty bounded closed and convex and if E is uniformly convex, then $A_K(\{x_t\})$ consists of a single point.

We need the following lemma to prove our next theorem.

Lemma 2 [22, Lem. 3.4]. *Let E be a Banach space with uniformly normal structure. Then for every bounded family $\{x_t\}_{t\in G}$ of elements of E, there exists y in $\overline{\text{co}}(\{x_t : t \in G\})$ such that*

$$\overline{\lim_t} \|x_t - y\| \le \tilde{N}(E) A(\{x_t\}), \tag{30}$$

where $\overline{\text{co}}(D)$ is the closure of the convex hull of $D \subseteq E$ and

$$A(\{x_t\}) = \lim_t \left(\sup\{\|x_i - x_j\| : t \le i, j \in G\}\right) \tag{31}$$

is the asymptotic diameter of $\{x_t\}$.

Now, we are in position to prove our next theorem.

Theorem 2. *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E, and $\mathcal{S} = \{T_s : s \in G\}$ a generalized uniformly Lipschitzian semigroup on K with $(\alpha+\beta) < \xi$, where $\xi > 1$ is the unique solution of (27), $\alpha = (a+b+c)/(1-b-c)$ and $\beta = (2b+2c)/(1-b-c)$. Suppose there is an x_0 in K such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in K such that $T_s z = z$ for all s in G.*

Proof. By induction, we define a sequence $\{x_n\}_0^\infty$ in K in the following manner:

$$x_{n+1} = A_K\left(\{T_t x_n\}_{t\in G}\right) \tag{32}$$

for $n = 0, 1, \ldots$, i.e., x_{n+1} is the unique point in K such that

$$\overline{\lim_t} \|T_t x_n - x_{n+1}\| = \inf_{y\in K} \overline{\lim_t} \|T_t x_n - y\|. \tag{33}$$

Write $r_n = r_K(\{T_t x_n\}_{t\in G})$. Then by Lemma 2, we have

$$\begin{aligned} r_n &= \overline{\lim_t} \|T_t x_n - x_{n-1}\| \\ &\le \tilde{N}(E)\cdot A\Big(\{T_t x_n\}_{t\in G}\Big) \\ &= \tilde{N}(E)\lim_t \Big(\sup\{\|T_i x_n - T_j x_n\| : t \le i,\, j\in G\}\Big) \\ &\le \tilde{N}(E)(\alpha+\beta)\cdot d(x_n), \end{aligned} \tag{34}$$

that is,

$$r_n \le (\alpha+\beta)\cdot \tilde{N}(E)d(x_n), \tag{35}$$

where $d(x_n) = \sup\{\|T_t x_n - x_n\| : t\in G\}$. We may assume that $d(x_n) > 0$ for all $n \ge 0$. Let $n \ge 0$ be fixed and let $\epsilon > 0$ be small enough. First, choose $j\in G$ such that

$$\|T_j x_{n+1} - x_{n+1}\| > d(x_{n+1}) - \epsilon \tag{36}$$

and then choose s_0 in G so large that

$$\|T_s x_n - x_{n+1}\| < r_n + \epsilon \tag{37}$$

and

$$\|T_s x_n - T_j x_{n+1}\| \le \alpha\|T_{s-j} x_n - x_{n+1}\| + \beta\|T_j x_n - x_n\| \le (\alpha+\beta)(r_n+\epsilon) \tag{38}$$

for all $s \ge s_0$. It, then, follows that

$$\Big\|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\Big\| \le (\alpha+\beta)(r_n+\epsilon)\left(1-\delta_E\left(\frac{d(x_{n+1})-\epsilon}{(\alpha+\beta)(r_n+\epsilon)}\right)\right) \tag{39}$$

for $s \ge s_0$ and, hence,

$$\begin{aligned} r_n &\le \overline{\lim_s}\Big\|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\Big\| \\ &\le (\alpha+\beta)(r_n+\epsilon)\left(1-\delta_E\left(\frac{d(x_{n+1})-\epsilon}{(\alpha+\beta)(r_n+\epsilon)}\right)\right). \end{aligned} \tag{40}$$

Taking the limit as $\epsilon \longrightarrow 0$, we get

$$r_n \le (\alpha+\beta)\cdot r_n\left(1-\delta_E\left(\frac{d(x_{n+1})}{(\alpha+\beta)r_n}\right)\right) \tag{41}$$

which together with (35) leads to the conclusion

$$d(x_{n+1}) \le (\alpha+\beta)^2\tilde{N}(E)\delta_E^{-1}\Big(1-\frac{1}{(\alpha+\beta)}\Big)d(x_n). \tag{42}$$

Hence,

$$d(x_n) \le Ad(x_{n-1}) \le A^n d(x_0), \tag{43}$$

where $A = (\alpha+\beta)^2\tilde{N}(E)\delta_E^{-1}(1-(1/(\alpha+\beta))) < 1$ by assumption. Noticing that

$$\begin{aligned} ||x_{n+1}-x_n|| &\le \overline{\lim_t}||T_t x_n - x_{n+1}|| + \overline{\lim_t}||T_t x_n - x_n|| \\ &\le r_n + d(x_n) \le 2d(x_n), \end{aligned} \tag{44}$$

we see from (43) that $\{x_n\}$ is a Cauchy sequence and, hence, strongly convergent. Let $z = \lim_n x_n$. Then we have, for each $s \in G$,

$$\begin{aligned} ||z - T_s z|| &\le ||z - x_n|| + ||T_s x_n - x_n|| + ||T_s x_n - T_s z|| \\ &\le (1+\alpha)||z - x_n|| + (1+\beta)\, d(x_n) \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{45}$$

This completes the proof. □

As a consequence of Theorem 2, we have the following result.

Corollary 1. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $T : K \longrightarrow K$ be a generalized uniformly Lipschitzian mapping with $(\alpha+\beta) < \xi$ (ξ is as in Theorem 2). Then T has a fixed point.*

If we take $b = c = 0$ in Theorem 2, then we have the following result from Theorem 2:

Corollary 2 [22, Thm. 3.5]**.** *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E, and $\mathcal{S} = \{T_s : s \in G\}$ a uniformly k-Lipschitzian semigroup on K with $k < \xi$, where $\xi > 1$ is the unique solution of (27). Suppose there is an x_0 in K such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in K such that $T_s z = z$ for all s in G.*

4. Some applications.

Since a Hilbert space H is 2-uniformly convex and the following equality holds:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{46}$$

for all x, y in H and $\lambda \in [0,1]$.

By Theorem 1 and (46), we immediately obtain the following:

Corollary 3. *Let E be a nonempty closed convex subset of a Hilbert space H, X be an l_G-invariant subspace of $m(G)$ containing constants which has left invariant submean μ, and $\mathcal{S} = \{T_s : s \in G\}$ be a generalized Lipschitzian semigroup on K. Suppose that there exists an x_0 in K such that $\{T_s x_0 : s \in G\}$ is a generalized Lipschitzian semigroup on K. Suppose that there exists an x_0 in K such that $\{T_s x_0 : s \in G\}$ is bounded and that for every u, v in K, then the function f on G defined by*

$$f(t) = \|T_t u - v\|^2, \quad t \in G \tag{47}$$

and the function g on G defined by

$$g(t) = 2(\alpha_t^2 + \beta_t^2), \quad t \in G \tag{48}$$

belong to X. Then, if $\{\mu_t(\alpha_t^2 + \beta_t^2)\} < 1$, where $\alpha_t = (a_t + b_t + c_t)/(1 - b_t - c_t)$ and $\beta_t = (2b_t + 2c_t)/(1 - b_t - c_t)$, there exists z in K such that $T_s z = z$ for all s in G.

If $1 < p \le 2$, then we have for all x, y in L^p and $\lambda \in [0,1]$

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)(p-1)\|x-y\|^2 \tag{49}$$

(the inequality (49) is contained in [12, 20]).

Assume that $2 < p < \infty$ and t_p is the unique zero of the function $g(x) = -x^{p-1} + (p-1)x + p - 2$ in the interval $(1,\infty)$. Let

$$c_p = (p-1)(1+t_p)^{2-p} = \frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}. \tag{50}$$

Then we have the following inequality

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)\cdot c_p \cdot \|x-y\|^p \tag{51}$$

for all x, y in L^p and $\lambda \in [0,1]$. (The inequality (51) is essentially due to Lim [11].)

By Theorem 1 and inequality (49) and (51), we immediately obtain the following result.

Corollary 4. *Let K be a closed convex subset of an L^p space, $1 < p < \infty$, X be an l_G-invariant subspace of $m(G)$ containing constants which has a left invariant submean μ, and $\mathcal{S} = \{T_s : s \in G\}$ be a generalized Lipschitzian semigroup on K. Suppose that $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in K$ and that for every u, v in K, the functions f and g on G defined as in Theorem 1 belong to X. If $2\mu_s(\alpha_s^2 + \beta_s^2) < p$ when $1 < p \le 2$ and $2^{p-1}\mu_s(\alpha_s^{p-1} + \beta_s^{p-1}) < 1 + c_p$ when $p > 2$, where $\alpha_s = (a_s + b_s + c_s)/(1 - b_s - c_s)$ and $\beta_s = (2b_s + 2c_s)/(1 - b_s - c_s)$, then there exists $z \in K$ such that $T_s z = z$ for all $s \in G$.*

Let H^p, $1 < p < \infty$, denote the Hardy space [5] of all functions x analytic in the unit disk $|z| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r \to 1^-}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|x(re^{i\theta})\right|^p d\theta\right)^{1/p} < \infty. \tag{52}$$

Now, let Ω be an open subset of $\mathbb{R}^n$. Denote by $H^{k,p}(\Omega)$, $k \ge 0$, $1 < p < \infty$, the Sobolev space [1, p. 149] of distribution x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = a_1 + \cdots + \alpha_n \le k$ equipped with the norm

$$\|x\| = \left(\sum_{|\alpha| \le k}\int_\Omega |D^\alpha x(\omega)|^p d\omega\right)^{1/p}. \tag{53}$$

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \wedge$, be a sequence of positive measure spaces, where index set $\wedge$ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < \infty$ and $q = \max\{2, p\}$ [13], the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \wedge\}$ equipped with the norm

$$\|x\| = \left(\sum_{\alpha \in \wedge}(\|x_\alpha\|_{p,\alpha})^q\right)^{1/q}, \tag{54}$$

where $\|\cdot\|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha,\Sigma_\alpha,\mu_\alpha)$.

Finally, let $L_p = (S_1,\Sigma_1,\mu_1)$ and $L_q = (S_2,\Sigma_2,\mu_2)$, where $1 < p < \infty$, $q = \max\{2,p\}$ and (S_i,Σ_i,μ_i) are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [4, III.2.10] of all measurable L_p-value function x on S_2 such that

$$\|x\| = \left(\int_{S_2} (\|x(S)\|_p)^q \mu_2(ds)\right)^{1/q}. \tag{55}$$

These spaces are q-uniformly convex with $q = \max\{2,p\}$ [16, 19] and the norm in these spaces satisfies

$$\|\lambda x + (1-\lambda)y\|^q \le \lambda\|x\|^q + (1-\lambda)\|y\|^q - d\cdot w_q(\lambda)\cdot\|x-y\|^q \tag{56}$$

with a constant

$$d = d_p = \begin{cases} \frac{p-1}{8} & \text{for } 1 < p \le 2 \\ \frac{1}{p\cdot 2^p} & \text{for } 2 < p < \infty. \end{cases} \tag{57}$$

Now, from Theorem 1, we have the following result.

Corollary 5. *Let K be a closed convex subset of the space E, where $E = H^p$, or $E = H^{k,p}(\Omega)$, or $E = L_{q,p}$, or $E = L_q(L_p)$, and $1 < p < \infty$, $q = \max\{2,p\}$, $k \ge 0$, X be an l_G-invariant subspace of $m(G)$ containing constants which has a left invariant submean μ, and $\mathcal{S} = \{T_s : s \in G\}$ be a generalized Lipschitzian semigroup on K. Suppose that $\{T_s x_0 : s \in G\}$ is bounded for some x_0 in K and that for every u,v in K, the functions f and g on G defined as in Theorem 1 belong to X. If $2^{q-1}\mu_s(\alpha_s^q + \beta_s^q) < 1+d$, where $\alpha_s = (a_s+b_s+c_s)/(1-b_s-c_s)$ and $\beta_s = (2b_s+2c_s)/(1-b_s-c_s)$, then there exists $z \in K$ such that $T_s z = z$ for all $s \in G$.*

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