NONWANDERING SETS OF MAPS ON THE CIRCLE

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Abstract. Let $f$ be a continuous map of the circle $S^1$ into itself. And let $R(f), \Lambda(f), \Gamma(f),$ and $\Omega(f)$ denote the set of recurrent points, $\omega$-limit points, $\gamma$-limit points, and nonwandering points of $f$, respectively. In this paper, we show that each point of $\Omega(f) \setminus R(f)$ is one-side isolated, and prove that

1. $\Omega(f) \setminus \Gamma(f)$ is countable and
2. $\Lambda(f) \setminus \Gamma(f)$ and $R(f) \setminus \Gamma(f)$ are either empty or countably infinite.

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1. Introduction. Let $I$ be the unit interval, $S^1$ the circle, and $X$ a topological space. And let $C^0(X,X)$ denote the set of continuous maps from $X$ into itself. For any $f \in C^0(X,X)$, let $P(f), R(f), \Lambda(f), \Gamma(f),$ and $\Omega(f)$ denote the set of periodic points, recurrent points, $\omega$-limit points, $\gamma$-limit points and nonwandering points of $f$, respectively.

For any $f \in C^0(I,I)$, in 1980, Z. Nitecki [6] has proved that each point of $\Omega(f) \setminus \overline{P(f)}$ is isolated in $\Omega(f)$ if $f$ is piecewise monotone and is not flat on any subinterval of $I$. In 1984, J. C. Xiong [7] has proved that each point of $\Omega(f) \setminus \overline{P(f)}$ is one-side isolated in $\Omega(f)$, for a continuous self map of interval $I$. And, in 1988, J. C. Xiong [9] also showed that $\Omega(f) \setminus \Gamma(f)$ is countable and that $\Lambda(f) \setminus \Gamma(f)$ and $\overline{P(f)} \setminus \Gamma(f)$ are either empty or countably infinite.

In this paper, we obtain the following similar results for maps of the circle:

**Theorem 1.1.** Let $f \in C^0(S^1,S^1)$. Then each point of $\Omega(f) \setminus \overline{R(f)}$ is one-side isolated in $\Omega(f)$.

**Theorem 1.2.** Let $f \in C^0(S^1,S^1)$. Then

1. $\Omega(f) \setminus \Gamma(f)$ is countable.
2. $\Lambda(f) \setminus \Gamma(f)$ and $\overline{R(f)} \setminus \Gamma(f)$ are either empty or countably infinite.

2. Preliminaries and definitions. Let $X$ be a compact metric space and $f \in C^0(X,X)$. For any positive integer $n$, we define $f^n$ inductively by $f^1 = f$ and $f^{n+1} = f \circ f^n$. Let $f^0$ denote the identity map of $X$. The forward orbit $\text{Orb}(x)$ of $x \in X$ is the set $\{f^k(x) \mid k = 0, 1, 2, \ldots\}$. Usually, the forward orbit of $x$ is simply called the orbit of $x$.

A point $x \in X$ is called a **periodic point** of $f$ if, for some positive integer $n, f^n(x) = x$. The period of $x$ is the least such integer $n$. We denote the set of periodic points of $f$ by $P(f)$. A point $x \in X$ is called a **recurrent point** of $f$ if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to x$. We denote the set of recurrent
points of $f$ by $R(f)$. A point $x \in X$ is called a nonwandering point of $f$ if, for every neighborhood $U$ of $x$, there exists a positive integer $m$ such that $f^m(U) \cap U \neq \emptyset$. We denote the set of nonwandering points of $f$ by $\Omega(f)$.

A point $y \in X$ is called an $\omega$-limit point of $x$ if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. We denote the set of $\omega$-limit points of $x$ by $\omega(x)$. Define $\Lambda(f) = \bigcup_{x \in X} \omega(x)$. A point $y \in X$ is called an $\alpha$-limit point of $x$ if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{y_i\}$ of points such that $f^{n_i}(y_i) = x$ and $y_i \to y$. The symbol $\alpha(x)$ denotes the set of $\alpha$-limit points of $x$. A point $y \in X$ is called a $\gamma$-limit point of $x$ if $y \in \omega(x) \cap \alpha(x)$. The symbol $\gamma(x)$ denotes the set of $\gamma$-limit points of $x$ and $\Gamma(f) = \bigcup_{x \in X} \gamma(x)$.

Let $R$ be the set of reals and $Z$ be the set of integers. Formally, we think of the circle $S^1$ as $R/Z$ and use $\pi: R \to R/Z$ to denote the canonical projection. In fact, the map $\pi: R \to S^1$ is an example of a covering map since it wraps $R$ around $S^1$ without doubling back (i.e., without critical points). To study the dynamics of the circle map, it is helpful to use a lifting. Let $f$ be a continuous map on the circle. We say that a continuous map $F$ from $R$ into itself is a lifting of $f$ if $f \circ \pi = \pi \circ F$. We use the following notations throughout this paper.

Let $a, b \in S^1$ with $a \neq b$, and let $A \in \pi^{-1}(a), B \in \pi^{-1}(b)$ with $|A - B| < 1$ and $A < B$. Then we write $\pi((A, B)], \pi([A, B]), \pi([A, B])$ and $\pi((A, B))$ to denote the open, closed, and half-open arcs from $a$ counterclockwise to $b$, respectively, and we denote it by $(a, b), [a, b], (a, b)$, and $(a, b)$. For $x, y \in [a, b]$ with $a \neq b$, let $X \in \pi^{-1}(x), Y \in \pi^{-1}(y)$ with $X, Y \in [A, B]$, then we define for $x, y \in [a, b], x > y$ if and only if $X > Y$. Let $C$ be a subset of a closed arc $[a, b]$, then we define $\sup C = \pi(\sup(\pi^{-1}(C) \cap [A, B]))$ and $\inf C = \pi(\inf(\pi^{-1}(C) \cap [A, B]))$.

In particular, for $a, b, c \in S^1, a < b < c$ means that $b$ lies in the open arc $(a, c)$, that is, $b \in (a, c)$.

Let $X$ be $I$ or $S^1$ and $Y \subset X$. Let $x \in Y$. A point $x \in X$ is said to be left-sided isolated (resp., right-sided isolated) in $Y$ if, for some $\epsilon > 0$, $(x - \epsilon, x) \cap Y = \emptyset$ (resp., $(x, x + \epsilon) \cap Y = \emptyset$). A point $x$ is said to be one-sided isolated in $Y$ if $x$ is either left-side or right-side isolated in $Y$, and a point $x$ which is both a right-sided and a left-sided isolated in $Y$ is said to be isolated in $Y$.

Let $x \in S^1$ and $f \in C^0(S^1, S^1)$ be given. Then we use the symbols $\omega_+(x)$ (resp., $\omega_-(x)$) to denote the set of all points $y \in S^1$ such that there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_1}(x) \to y$ and $y < \cdots < f^{n_2}(x) < f^{n_1}(x) < \cdots < y$. It is clear that if $x \notin P(f)$, then $\omega(x) = \omega_+(x) \cup \omega_-(x)$. Define $\Lambda_+(f) = \bigcup_{x \in S^1} \omega_+(x)$ and $\Lambda_-(f) = \bigcup_{x \in S^1} \omega_-(x)$.

Also, we use the symbols $\alpha_+(x)$ (resp., $\alpha_-(x)$) to denote the set of all points $y \in S^1$ such that there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{x_i\}$ of points such that $x_i \to y, f^{n_i}(x_i) = x$ for every $i > 0$ and $y < \cdots < x_i < \cdots < x_2 < x_1$ (resp., $x_1 < x_2 < \cdots < x_i < \cdots < y$). It is clear that if $x \notin P(f)$, then $\alpha(x) = \alpha_+(x) \cup \alpha_-(x)$.

Define $\gamma_+(x) = \omega_+(x) \cap \alpha_+(x)$ and $\gamma_-(x) = \omega_-(x) \cap \alpha_-(x)$. Also, we define $\Gamma_+(f) = \bigcup_{x \in S^1} \gamma_+(x)$ and $\Gamma_-(f) = \bigcup_{x \in S^1} \gamma_-(x)$.

Let $Y$ be an arc in $S^1$ and let $\overline{Y}$ denote the closure of $Y$ as usual. A point $y \in S^1$ is
called a right-sided (resp., left-sided) accumulation point of \( Y \) if, for any \( z \in S^1, (y, z) \cap Y = \phi \) (resp. \( (z, y) \cap Y = \phi \)).

The right-side closure \( \overline{Y}_+ \) (resp. left-side closure \( \overline{Y}_- \)) is the union of \( Y \) and the set of right-sided (resp. left-sided) accumulation points of \( Y \). A point which is both a right-sided and a left-sided accumulation point of \( Y \) is called a two-sided accumulation point of \( Y \).

3. Main results. The following lemmas are founded in [3].

**Lemma 3.1.** Let \( f \in C^0(S^1, S^1) \) and \( x \in \Omega (f) \). Then we have \( x \in \alpha(x) \).

**Lemma 3.2.** Let \( f \in C^0(S^1, S^1) \) and \( I = [a, b] \) be an arc for some \( a, b \in S^1 \) with \( a \neq b \), and let \( I \cap P(f) = \phi \).

(a) Suppose that there exists \( x \in I \) such that \( f(x) \in I \) and \( x < f(x) \). Then
   (i) if \( y \in I, x < y, \) and \( f(y) \notin [y, b] \), then \([x, y]f\)-covers \([f(x), b]\), and
   (ii) if \( y \in I, y < x, \) and \( f(y) \notin [y, b] \), then \([y, x]f\)-covers \([f(x), b]\).

(b) Suppose that there exists \( x \in I \) such that \( f(x) \in I \) and \( x > f(x) \). Then
   (i) if \( y \in I, x < y, \) and \( f(y) \notin [a, y] \), then \([x, y]f\)-covers \([a, f(x)]\), and
   (ii) if \( y \in I, y < x, \) and \( f(y) \notin [a, y] \), then \([y, x]f\)-covers \([a, f(x)]\).

**Lemma 3.3.** Let \( f \in C^0(S^1, S^1) \). Then we have

\[
P(f) \subset R(f) \subset \Gamma (f) \subset \overline{R(f)} \subset A(f) \subset \Omega (f) \subset CR(f). \tag{1}
\]

The following lemma is due to [5]

**Lemma 3.4.** Let \( f \in C^0(S^1, S^1) \), and let \( K \subset S^1 \) with \( f(K) \subset K \). If \( x \in \Omega (f) \setminus K \), then \( f^n(x) \notin K \) for any \( n \geq 1 \).

The idea of the proof of the following lemma is due to [7].

**Lemma 3.5.** Let \( f \in C^0(S^1, S^1) \), and let \( K \subset S^1 \) have only finitely many connected components and \( f(K) = K \). Then we have \( \overline{K} \setminus K \subset P(f) \).

**Proof.** By continuity of \( f \), we have \( f(\overline{K}) \subset \overline{f(K)} \). And by the compactness of \( K, f(\overline{K}) \subset S^1 \) is closed. Thus, \( f(K) \subset \overline{f(K)} = f(\overline{K}) \). Therefore, \( f(\overline{K}) = f(K) = K \). Hence, for each \( x \in \overline{K} \setminus K \), there exists \( x' \in \overline{K} \setminus K \) such that \( f(x') = x \), i.e., \( f(\overline{K} \setminus K) = \overline{K} \setminus K \). By the finiteness of \( \overline{K} \setminus K, \overline{K} \setminus K \subset P(f) \).

**Proposition 3.6.** Let \( f \in C^0(S^1, S^1) \). Suppose that \( x \in \Omega (f) \setminus \overline{R(f)} \).

1. If \( x \in \alpha_+(x) \), then there exists \( z \in S^1 \) such that \( f^i(z, x) \cap (z, x) = \phi \) for all \( i \geq 1 \).
2. If \( x \in \alpha_-(x) \), then there exists \( u \in S^1 \) such that \( f^i(x, u) \cap (x, u) = \phi \) for all \( i \geq 1 \).

**Proof.** We only need to prove part (1). There exists \( a, b \in S^1 \) such that \( x \in (a, b) \) and \( (a, b) \cap \text{Orb}(x) = \phi \). Let \( V = (a, x) \) and let \( W = \bigcup_{i=0}^{\infty} f^i(V) \). Then \( x \in \overline{W} \). Since \( x \in \alpha_+(x) \), there exist a positive integer \( m \) and a point \( y \in (x, b) \) such that \( f^m(y) = x \).

By Lemma 3.2,

\[
[y, x]f^m\text{-covers } [a, x]. \tag{2}
\]
We claim that \( x \notin W \). To show this, suppose that \( x \in W \). Then there exist a positive integer \( j \) and a point \( x_0 \in (a,x) \) such that \( f^j(x_0) = x \). By Lemma 3.2,

\[
[x_0,x] f^j \text{-covers } [x,b].
\]

In particular, \([x_0,x] f^j \text{-covers } [x,y]\).

By (2),

\[
[x_0,x] f^j \text{-covers } [x_0,y].
\]

Thus,

\[
[x_0,x] f^{j+m} \text{-covers itself},
\]

and, hence, \( f^{j+m} \) has a periodic point in \((a,b)\), a contradiction. Hence, we have \( x \in \overline{W} \setminus W \).

Assume that the proposition is false, i.e., for each \( z \in (a,x) \), there is some \( i \geq 1 \) such that \( (z,x) \cap f^i(z,x) \neq \emptyset \). Note that \( V \subset f(W) \). Because, for each \( y' \in V \), there is some \( i \geq 1 \) such that \( (y',x) \cap f^i(y',x) \neq \emptyset \). There exists \( x_0 \in (y',x) \) such that \( f^i(x_0) \in (y',x) \). By Lemma 3.2, either

\[
[x_0,x] f^i \text{-covers } [f^i(x_0),b] \quad \text{or} \quad [x_0,x] f^i \text{-covers } [a,f^i(x_0)].
\]

Particularly, either

\[
[x_0,x] f^i \text{-covers } [x,b] \quad \text{or} \quad [x_0,x] f^i \text{-covers } [a,f^i(x_0)].
\]

If

\[
[x_0,x] f^i \text{-covers } [x,b],
\]

then

\[
[x_0,x] f^j \text{-covers } [x,y].
\]

By (2),

\[
[x,y] f^m \text{-covers } [x_0,x].
\]

Hence,

\[
[x_0,x] f^{j+m} \text{-covers itself}.
\]

Thus, \( f^{j+m} \) has a periodic point in \((a,b)\). This is a contradiction. Therefore,

\[
[x_0,x] f^j \text{-covers } [a,f^i(x_0)].
\]

Thus, \( y' \in f^i(x_0,x) \subset f^i(V) \subset f(W) \) since \( y' \in (a,f^i(x_0)) \). Thus, for each \( i = 1,2,3,\ldots,l-1 \), \( f^i(V) \cap f^{l+i}(V) \neq \emptyset \), and \( f^{l+i}(V) \cap f^{2l+i}(V) \neq \emptyset \), … Therefore, \( U_i = \bigcup_{m=0}^{\infty} f^{mi+i}(V) \) is connected and \( W = \bigcup_{l=0}^{l-1} U_l \) has only finitely many connected components. Now, by Lemma 3.5, \( x \in \overline{W} \setminus W \subset P(f) \). This is in contradiction with the assumption of this proposition. \( \square \)
The following theorem follows immediately from the proposition.

**Theorem 3.7.** Let \( f \in C^0(S^1, S^1) \). Then each point of \( \Omega(f) \setminus \overline{R(f)} \) is one-side isolated in \( \Omega(f) \).

**Corollary 3.8.** Let \( f \in C^0(S^1, S^1) \). Then \( \Omega(f) \setminus \overline{R(f)} \) is countable which is nowhere dense in \( S^1 \).

The following proposition is found in [1].

**Proposition 3.9.** Let \( f \in C^0(S^1, S^1) \). Then we have

1. \( \overline{R(f)}_+ \setminus R(f) \subset \Lambda(f)_+ \).
2. \( \overline{R(f)}_- \setminus R(f) \subset \Lambda(f)_- \).

**Proposition 3.10.** Let \( f \in C^0(S^1, S^1) \). Then we have \( \overline{R(f)}_+ \cap \overline{R(f)}_- \setminus R(f) \subset \Gamma(f) \).

**Proof.** If \( P(f) = \phi \), then we have the desired results since \( \overline{R(f)} = \Gamma(f) \) [2]. Suppose that \( P(f) \neq \phi \). Let \( z \in \overline{R(f)}_+ \cap \overline{R(f)}_- \setminus R(f) \). Then there exist \( a, b \in S^1 \) with \( a < b \) such that \( z \in (a, b) \) and \( (a, b) \cap \text{Orb}(z) = \phi \). By Proposition 3.9, \( z \in \Lambda(f)_+ \cap \Lambda(f)_- \). Then there exist \( y_1, y_2 \) such that \( a < y_1 < z < y_2 < b \) with \( z \in \omega(y_1) \cap \omega(y_2) \). Since \( P(f) = \overline{R(f)} \) [4], \( z \in \overline{P(f)}_+ \cap \overline{P(f)}_- \setminus P(f) \). Then there exists \( u_i \) of periodic point of \( f \) with \( a < y_1 < u_1 < u_2 < \cdots < z \) and \( u_i \to z \). Let \( p_i \) be the period of \( u_i \) with respect to \( f \). Then \( f^{p_i}(u_i) = u_i \) for all \( i \geq 1 \). Then either \( [u_i, z] f^{p_i} \)-covers \( [a, u_i] \) or \( [u_i, z] f^{p_i} \)-covers \( [u_i, b] \).

We may assume that, for infinitely many \( i \), either

\[
[u_i, z] f^{p_i} \text{-covers } [a, u_i] \quad \text{or} \quad [u_i, z] f^{p_i} \text{-covers } [u_i, b].
\]

Then we consider two cases.

**Case I.** \( [u_i, z] f^{p_i} \)-covers \( [a, u_i] \) for infinitely many \( i \). There exists \( z_i \in [u_i, z] \) such that \( f^{p_i}(z_i) = y_1 \). Since \( u_i \to z, z_i \to z \). Thus, \( z \in \alpha(y_1) \) and, hence, \( z \in \omega(y_1) \cap \alpha(y_1) \subset \Gamma(f) \).

**Case II.** \( [u_i, z] f^{p_i} \)-covers \( [u_i, b] \) for infinitely many \( i \). There exists \( z_i' \in [u_i, z] \) such that \( f^{p_i}(z_i') = y_1 \). Since \( u_i \to z, z_i' \to z \). Thus, \( z \in \alpha(y_2) \) and, hence, \( z \in \omega(y_2) \cap \alpha(y_2) \subset \Gamma(f) \).

\[ \square \]

The idea of the proof of the following lemma is due to [8].

**Lemma 3.11.** Let \( f \in C^0(S^1, S^1) \) and \( Y \subset S^1 \). Then \( \overline{Y} \setminus (\overline{Y} \cap \overline{Y}^-) \) is countable.

**Proof.** For each \( y \in \overline{Y}^+ \setminus \overline{Y}^- \), there is some \( u_y \in S^1 \) such that \( (u_y, y) \cap Y = \phi \). The family of \( \{(u_y, y) \mid y \in \overline{Y}^+ \setminus \overline{Y}^-\} \) is countable because it is disjoint. Hence, \( \overline{Y}^+ \setminus \overline{Y}^- \) is countable. Similarly, \( \overline{Y}^- \setminus \overline{Y}^+ \) is also countable. Therefore,

\[ \overline{Y} \setminus (\overline{Y}^+ \cap \overline{Y}^-) = (\overline{Y}^+ \setminus \overline{Y}^-) \cup (\overline{Y}^- \setminus \overline{Y}^+) \]

is countable.

\[ \square \]

**Theorem 3.12.** Let \( f \in C^0(S^1, S^1) \). Then

1. \( \Omega(f) \setminus \Gamma(f) \) is countable.
2. \( \Lambda(f) \setminus \Gamma(f) \) and \( \overline{R(f)} \setminus \Gamma(f) \) are either empty or countably infinite.
Proof. (1) We know that $R(f) \setminus (R(f)_+ \cap R(f)_-)$ is countable by Lemma 3.11. By Proposition 3.10, $R(f) \setminus \Gamma(f)$ is also countable. By Corollary 3.8, $\Omega(f) \setminus R(f)$ is countable. Hence, $\Omega(f) \setminus \Gamma(f)$ is countable.

(2) It is easy to prove that $f(\omega(x)) = \omega(x)$ and $f(R(f)) = R(f)$ for $x \in S^1$. Hence, $f(\Lambda(f)) = \Lambda(f)$. Suppose that $\Lambda(f) \setminus \Gamma(f) \neq \emptyset$ (resp., $R(f) \setminus \Gamma(f) \neq \emptyset$). Then we take $z_1 \in \Lambda(f) \setminus \Gamma(f)$ (resp., $z_1 \in R(f) \setminus \Gamma(f)$). We can take $z_2 \in \Lambda(f) \setminus \Gamma(f)$ (resp., $z_2 \in R(f) \setminus \Gamma(f)$) such that $z_1 = f(z_2)$. Continuing this process, we can take $z_i \in \Lambda(f) \setminus \Gamma(f)$ (resp., $z_i \in R(f) \setminus \Gamma(f)$) such that $z_i = f(z_{i+1})$ for all $i = 1, 2, \ldots$. Since $z_i \notin f$ for all $i \geq 1$, the points $z_1, z_2, \ldots$ are pairwise disjoint. Hence, $\Lambda(f) \setminus \Gamma(f)$ (resp., $R(f) \setminus \Gamma(f)$) is infinite and, hence, $\Lambda(f) \setminus \Gamma(f)$ (resp., $R(f) \setminus \Gamma(f)$) is countably infinite.

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