A CHARACTERIZATION OF RANDOM APPROXIMATIONS

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ABSTRACT. By using Hahn-Banach theorem, a characterization of random approximations is obtained.

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1. Introduction and preliminaries. Random approximation theory is a lively and fascinating field of research lying at the intersection of approximation theory and probability theory. It has received much attention for the past two decades after the publication of a survey article by Bharucha-Reid [4] in 1976. For more details, see [1, 2, 3, 5, 6, 7, 8, 9] and references therein. Random approximation theorems are required for the theory of random equations. The aim of this note is to obtain a characterization of random approximation via the Hahn-Banach theorem. Let (Ω, Σ) be a measurable space with Σ a sigma algebra of subsets of Ω . Let X be a normed space and M be a nonempty subset of X . A map $T : \Omega \times M \rightarrow X$ is called a random operator if for each fixed $x \in M$, the map $T(\cdot, x) : \Omega \rightarrow X$ is measurable. Let $B_r(x) := \{z \in X : \|z - x\| \leq r\}$ and $\delta(M, x) := \inf_{u \in M} \|x - u\|$. In the sequel, cl , int , and X' stand for the closure, interior, and normed dual of X .

In our proof, we use the following geometric version of the Hahn-Banach theorem regarding the separation of convex sets: Let A and B be two disjoint convex sets in a normed space X . Moreover, assume that A is open. Then, there is an $f \in X'$ and a real number c such that $\text{Re } f(x) > c$ for $x \in A$, and $\text{Re } f(x) \leq c$ for $x \in B$.

2. The results

THEOREM. *Let M be a nonempty convex subset of a complex normed space X , $T : \Omega \times M \rightarrow X$ be a random operator, and $\xi : \Omega \rightarrow M$ be a measurable map such that $T(\omega, \xi(\omega)) \notin \text{cl}(M)$. Then ξ is a random best approximation for T , i.e., $\|\xi(\omega) - T(\omega, \xi(\omega))\| = \delta(M, T(\omega, \xi(\omega)))$ if and only if there exists $f \in X'$ with the following properties:*

- (a) $\|f\| = 1$,
- (b) $f(T(\omega, \xi(\omega)) - \xi(\omega)) = \|T(\omega, \xi(\omega)) - \xi(\omega)\|$, and
- (c) $\text{Re } f(x - \xi(\omega)) \leq 0$ for all $x \in M$.

PROOF. Necessity: Assume that $\|\xi(\omega) - T(\omega, \xi(\omega))\| = \delta(M, T(\omega, \xi(\omega)))$. Then M and $\text{int}(B_r(T(\omega, \xi(\omega))))$, where $r := \|T(\omega, \xi(\omega)) - \xi(\omega)\|$, are disjoint convex sets. By the separation theorem, there is an $f_{\xi(\omega)} \in X'$ and $c \in \mathbb{R}$ such that,

$$\operatorname{Re} f_{\xi(\omega)}(x) \leq c \quad \text{for all } x \in M \quad (1)$$

and

$$\operatorname{Re} f_{\xi(\omega)}(y) > c \quad \text{for all } y \in \operatorname{int} \left(B_r \left(T(\omega, \xi(\omega)) \right) \right). \quad (2)$$

The continuity of $f_{\xi(\omega)}$ implies that,

$$\operatorname{Re} f_{\xi(\omega)}(y) \geq c \quad \text{for all } y \in B_r \left(T(\omega, \xi(\omega)) \right). \quad (3)$$

Since $\xi(\omega) \in M \cap B_r(T(\omega, \xi(\omega)))$, $\operatorname{Re} f_{\xi(\omega)}(\xi(\omega)) = c$. Also, since $T(\omega, \xi(\omega)) \in \operatorname{int} B_r(T(\omega, \xi(\omega)))$, it follows that,

$$\beta := \operatorname{Re} f_{\xi(\omega)} \left(T(\omega, \xi(\omega)) \right) - c = \operatorname{Re} f_{\xi(\omega)} (T(\omega, \xi(\omega)) - \xi(\omega)) > 0. \quad (4)$$

Let $f = \beta^{-1} r f_{\xi(\omega)}$. This implies that

$$\begin{aligned} \operatorname{Re} f(T(\omega, \xi(\omega)) - \xi(\omega)) &= \operatorname{Re} \beta^{-1} r f_{\xi(\omega)} (T(\omega, \xi(\omega)) - \xi(\omega)) \\ &= \beta^{-1} r \operatorname{Re} f_{\xi(\omega)} (T(\omega, \xi(\omega)) - \xi(\omega)) \\ &= r \\ &= \| (T(\omega, \xi(\omega)) - \xi(\omega)) \|. \end{aligned} \quad (5)$$

It further implies that $\|f\| \geq 1$.

Suppose that $\|f\| > 1$. Then there would exist an $h \in X$, with $\|h\| < 1$, such that $f(h)$ is real and $f(h) > 1$. For $y = T(\omega, \xi(\omega)) - rh$, we have,

$$\operatorname{Re} f_{\xi(\omega)}(y) = \operatorname{Re} \left[f_{\xi(\omega)} \left(T(\omega, \xi(\omega)) \right) - r f_{\xi(\omega)}(h) \right] = (c + \beta) - \beta f(h) < c. \quad (6)$$

Since $y \in B_r(T(\omega, \xi(\omega)))$, the above inequality contradicts inequality (3). Hence, $\|f\| = 1$. As $\|f\| = 1$, it follows that $|f(T(\omega, \xi(\omega)) - \xi(\omega))| \leq \|T(\omega, \xi(\omega)) - \xi(\omega)\|$. This and equality (5) imply that $f(T(\omega, \xi(\omega)) - \xi(\omega)) = \|T(\omega, \xi(\omega)) - \xi(\omega)\|$. Finally, from inequalities (2) and (3), we obtain,

$$\operatorname{Re} f_{\xi(\omega)}(x - \xi(\omega)) = \operatorname{Re} f_{\xi(\omega)}(x) - \operatorname{Re} f_{\xi(\omega)}(\xi(\omega)) \leq 0, \quad (7)$$

for $x \in M$. Since $f = \beta^{-1} r f_{\xi(\omega)}$, where $\beta^{-1} r > 0$,

$$\operatorname{Re} f(x - \xi(\omega)) = \operatorname{Re} \beta^{-1} r f_{\xi(\omega)}(x - \xi(\omega)) \leq 0. \quad (8)$$

Sufficiency: Let M be a nonempty set in a complex normed space X and let $\xi : \Omega \rightarrow M$ be a measurable map. Assume that there is an $f \in X'$ satisfying (a), (b), and (c).

For each $x \in M$,

$$\begin{aligned} \operatorname{Re} f(T(\omega, \xi(\omega)) - x) &\leq |f(T(\omega, \xi(\omega)) - x)| \\ &\leq \|f\| \|T(\omega, \xi(\omega)) - x\| \\ &= \|T(\omega, \xi(\omega)) - x\|. \end{aligned} \quad (9)$$

It further implies that

$$\begin{aligned}
 \|T(\omega, \xi(\omega)) - x\| &\geq \operatorname{Re} f(T(\omega, \xi(\omega)) - x) \\
 &= \operatorname{Re} f(T(\omega, \xi(\omega)) - \xi(\omega)) - \operatorname{Re} f(x - \xi(\omega)) \\
 &\geq \operatorname{Re} f(T(\omega, \xi(\omega)) - \xi(\omega)) \\
 &= \|T(\omega, \xi(\omega)) - \xi(\omega)\|.
 \end{aligned} \tag{10}$$

Hence, $\|T(\omega, \xi(\omega)) - \xi(\omega)\| = \delta(M, T(\omega, \xi(\omega)))$. $\square$

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