ON A CLASS OF UNIVALENT FUNCTIONS

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ABSTRACT. We consider the class of univalent functions defined by the conditions $f(z)/z \neq 0$ and $|(z/f(z))''| \leq \alpha, |z| < 1$, where $f(z) = z + \dots$ is analytic in $|z| < 1$ and $0 < \alpha \leq 2$.

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1. Introduction. Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f(z) \in A$ is said to be star-like in $|z| < r$ ($r \leq 1$) if and only if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad (|z| < r). \quad (1.2)$$

In [2], Nunokawa, Obradovic, and Owa proved the following theorem:

THEOREM A. *Let $f(z) \in A$ with $f(z) \neq 0$ for $0 < |z| < 1$ and let*

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 1, \quad (z \in E). \quad (1.3)$$

Then $f(z)$ is univalent in E .

For $0 < \alpha \leq 2$, let $S(\alpha)$ denote the class of functions $f(z) \in A$ which satisfy the conditions

$$f(z) \neq 0 \quad \text{for } 0 < |z| < 1 \quad (1.4)$$

and

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq \alpha, \quad (z \in E). \quad (1.5)$$

In this paper, we give an extension of Theorem A and obtain some results for the class $S(\alpha)$.

By virtue of a result due to Ozaki and Nunokawa [4], Obradovic et al. [3] considered a class of univalent functions.

2. A criterion for univalence

THEOREM 1. *Let $f(z) \in A$ with $f(z) \neq 0$ for $0 < |z| < 1$ and let $g(z) \in A$ be bounded*

in E and satisfy

$$m = \inf \left\{ \left| \frac{g(z_1) - g(z_2)}{z_1 - z_2} \right| : z_1, z_2 \in E \right\} > 0. \quad (2.1)$$

If

$$\left| \left(\frac{z}{f(z)} - \frac{z}{g(z)} \right)'' \right| \leq K, \quad (z \in E), \quad (2.2)$$

where

$$K = \frac{2m}{M^2} \quad \text{and} \quad M = \sup \{ |g(z)| : z \in E \}, \quad (2.3)$$

then $f(z)$ is univalent in E .

PROOF. If we put

$$h(z) = \left(\frac{z}{f(z)} - \frac{z}{g(z)} \right)'', \quad (2.4)$$

then the function $h(z)$ is analytic in E and, by integration from 0 to z , we get

$$\left(\frac{z}{f(z)} - \frac{z}{g(z)} \right)' = b_2 - a_2 + \int_0^z h(u) du \quad (2.5)$$

and

$$\frac{z}{f(z)} - \frac{z}{g(z)} = (b_2 - a_2)z + \int_0^z dv \int_0^v h(u) du, \quad (2.6)$$

where $f(z) = z + a_2 z^2 + \dots$ and $g(z) = z + b_2 z^2 + \dots$.

Thus, we have

$$f(z) = \frac{g(z)}{1 + (b_2 - a_2)g(z) + g(z)(\psi(z)/z)}, \quad (2.7)$$

where

$$\psi(z) = \int_0^z dv \int_0^v h(u) du. \quad (2.8)$$

Since

$$\left(\frac{\psi(z)}{z} \right)' = \frac{1}{z^2} \int_0^z u \psi''(u) du = \frac{1}{z^2} \int_0^z u h(u) du, \quad (2.9)$$

from (2.2) and (2.4), we get

$$\left| \left(\frac{\psi(z)}{z} \right)' \right| \leq \int_0^1 t |h(zt)| dt \leq \frac{K}{2}, \quad (2.10)$$

and so

$$\left| \frac{\psi(z_2)}{z_2} - \frac{\psi(z_1)}{z_1} \right| = \left| \int_{z_1}^{z_2} \left(\frac{\psi(z)}{z} \right)' dz \right| \leq \frac{K}{2} |z_2 - z_1| \quad (2.11)$$

for $z_1, z_2 \in E$ and $z_1 \neq z_2$.

If $z_1 \neq z_2$ then $g(z_1) \neq g(z_2)$ and it follows, from (2.7) and (2.11), that

$$\begin{aligned}
& |f(z_1) - f(z_2)| \\
&= \frac{\left| g(z_1) - g(z_2) + g(z_1)g(z_2) \left(\frac{\psi(z_2)}{z_2} - \frac{\psi(z_1)}{z_1} \right) \right|}{\left| 1 + (b_2 - a_2)g(z_1) + g(z_1)\frac{\psi(z_1)}{z_1} \right| \left| 1 + (b_2 - a_2)g(z_2) + g(z_2)\frac{\psi(z_2)}{z_2} \right|} \\
&> \frac{|g(z_1) - g(z_2)| - M^2 K \frac{|z_1 - z_2|}{2}}{\left| 1 + (b_2 - a_2)g(z_1) + g(z_1)\frac{\psi(z_1)}{z_1} \right| \left| 1 + (b_2 - a_2)g(z_2) + g(z_2)\frac{\psi(z_2)}{z_2} \right|} \geq 0.
\end{aligned} \tag{2.12}$$

Hence, $f(z)$ is univalent in E . $\square$

COROLLARY 1. Let $f(z) \in A$ with $f(z) \neq 0$ for $0 < |z| < 1$. If

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2, \quad (z \in E), \tag{2.13}$$

then $f(z)$ is univalent in E . The bound 2 in (2.13) is best possible.

PROOF. Setting $g(z) = z$ in Theorem 1, we conclude that $f(z)$ is univalent in E for $f(z)$ satisfying condition (2.13).

To show that the result is sharp, we consider

$$f(z) = \frac{z}{(1+z)^{2+\epsilon}}, \quad (\epsilon > 0). \tag{2.14}$$

Note that

$$\left| \left(\frac{z}{f(z)} \right)'' \right| = (2+\epsilon)(1+\epsilon)|1+z|^\epsilon, \quad (z \in E) \tag{2.15}$$

and $f'(1/(1+\epsilon)) = 0$. Hence, $f(z)$ is not univalent in E and the proof is complete. $\square$

From Corollary 1, we easily get

COROLLARY 2. Let

$$f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n} \in A \tag{2.16}$$

and

$$\sum_{n=2}^{\infty} n(n-1)|b_n| \leq 2. \tag{2.17}$$

Then $f(z)$ is univalent in E .

3. The class $S(\alpha)$. According to Corollary 1, all the functions in $S(\alpha)$ ($0 < \alpha \leq 2$) are univalent in E . Let the functions $f(z)$ and $g(z)$ be analytic in E . Then $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic in E , with $w(0) = 0$ and $|w(z)| < 1$ ($z \in E$), such that $f(z) = g(w(z))$ for $z \in E$.

For our next results, we need the following.

LEMMA 1 [5]. Let $f(z)$ and $g(z)$ be analytic in E with $f(0) = g(0)$. If $h(z) = zg'(z)$ is star-like in E and $zf'(z) < h(z)$, then

$$f(z) < g(z) = g(0) + \int_0^z \frac{h(t)}{t} dt. \quad (3.1)$$

THEOREM 2. Let $f(z) = z + a_2 z^2 + \dots \in S(\alpha)$ with $0 < \alpha \leq 2$. Then, for $z \in E$,

$$\left| \frac{z}{f(z)} - 1 \right| \leq |z| \left(|a_2| + \frac{\alpha}{2} |z| \right); \quad (3.2)$$

$$1 - |z| \left(|a_2| + \frac{\alpha}{2} |z| \right) \leq \operatorname{Re} \frac{z}{f(z)} \leq 1 + |z| \left(|a_2| + \frac{\alpha}{2} |z| \right); \quad (3.3)$$

$$|f(z)| \geq \frac{|z|}{1 + |a_2||z| + \frac{\alpha}{2}|z|^2}. \quad (3.4)$$

Equalities in (3.2), (3.3), and (3.4) are attained if we take

$$f(z) = \frac{z}{1 \pm az + \frac{\alpha}{2} z^2} \in S(\alpha), \quad (0 \leq a \leq \sqrt{2\alpha}). \quad (3.5)$$

PROOF. In view of (1.5), we have

$$z \left(\frac{z}{f(z)} \right)'' < \alpha z. \quad (3.6)$$

Applying the lemma to (3.6), we find that

$$\left(\frac{z}{f(z)} \right)' + a_2 < \alpha z. \quad (3.7)$$

By using a result of Hallenbeck and Ruscheweyh [1, Thm. 1], (3.7) gives

$$\frac{1}{z} \int_0^z \left[\left(\frac{t}{f(t)} \right)' + a_2 \right] dt < \frac{\alpha}{2} z, \quad (3.8)$$

i.e.,

$$\frac{z}{f(z)} = 1 - a_2 z + \frac{\alpha}{2} zw(z), \quad (3.9)$$

where $w(z)$ is analytic in E and $|w(z)| \leq |z|$ ($z \in E$) by Schwarz lemma.

Now, from (3.9), we can easily derive the inequalities (3.2), (3.3), and (3.4). $\square$

THEOREM 3. Let $f(z) \in S(\alpha)$ and have the form

$$f(z) = z + a_3 z^3 + a_4 z^4 + \dots. \quad (3.10)$$

(a) If $2/\sqrt{5} \leq \alpha \leq 2$, then $f(z)$ is star-like in $|z| < \sqrt{2/\alpha} \cdot 1/\sqrt[4]{5}$;

(b) If $\sqrt{3} - 1 \leq \alpha \leq 2$, then $\operatorname{Re} f'(z) > 0$ for $|z| < \sqrt{(\sqrt{3} - 1)/\alpha}$.

PROOF. If we put

$$p(z) = \frac{z^2 f'(z)}{f^2(z)} = 1 + p_2 z^2 + \dots, \quad (3.11)$$

then, by (1.5), we have

$$zp'(z) = -z^2 \left(\frac{z}{f(z)} \right)'' < \alpha z, \quad (3.12)$$

and it follows, from the lemma, that

$$p(z) < 1 + \alpha z, \quad (3.13)$$

which implies that

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq \alpha |z|^2, \quad (z \in E). \quad (3.14)$$

(a) Let $2/\sqrt{5} \leq \alpha \leq 2$ and

$$|z| < r_1 = \sqrt{\frac{2}{\alpha}} \cdot \frac{1}{\sqrt[4]{5}}. \quad (3.15)$$

Then, by (3.14), we have

$$\left| \arg \frac{z^2 f'(z)}{f^2(z)} \right| < \arcsin \frac{2}{\sqrt{5}}. \quad (3.16)$$

Also, from (3.2) in Theorem 2 with $a_2 = 0$, we obtain

$$\left| \frac{z}{f(z)} - 1 \right| < \frac{\alpha}{2} r_1^2, \quad (3.17)$$

and so

$$\left| \arg \frac{z}{f(z)} \right| < \arcsin \frac{1}{\sqrt{5}}. \quad (3.18)$$

Therefore, it follows, from (3.16) and (3.18), that

$$\left| \arg \frac{zf'(z)}{f(z)} \right| \leq \left| \arg \frac{z^2 f'(z)}{f^2(z)} \right| + \left| \arg \frac{z}{f(z)} \right| < \arcsin \frac{2}{\sqrt{5}} + \arcsin \frac{1}{\sqrt{5}} = \frac{\pi}{2} \quad (3.19)$$

for $|z| < r_1$. This proves that $f(z)$ is star-like in $|z| < r_1$.

(b) Let $\sqrt{3} - 1 \leq \alpha \leq 2$ and

$$|z| < r_2 = \sqrt{\frac{\sqrt{3}-1}{\alpha}}. \quad (3.20)$$

Then we have

$$\begin{aligned} \left| \arg f'(z) \right| &\leq \left| \arg \frac{z^2 f'(z)}{f^2(z)} \right| + 2 \left| \arg \frac{z}{f(z)} \right| < \arcsin(\alpha r_2^2) + 2 \arcsin\left(\frac{\alpha}{2} r_2^2\right) \\ &= \arcsin(\sqrt{3}-1) + 2 \arcsin\left(\frac{\sqrt{3}-1}{2}\right) = \frac{\pi}{2}. \end{aligned} \quad (3.21)$$

Thus, $\operatorname{Re} f'(z) > 0$ for $|z| < r_2$. □

COROLLARY 3. Let $f(z) \in S(\alpha)$ and have the form (3.10)

- (a) if $0 < \alpha \leq 2/\sqrt{5}$, then $f(z)$ is star-like in E ;
- (b) if $0 < \alpha \leq \sqrt{3}-1$, then $\operatorname{Re} f'(z) > 0$ for $z \in E$.

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