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s-POINT FINITE REFINABLE SPACES

SHELDON W. DAVIS, ELISE M. GRABNER, and GARY C. GRABNER

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ABSTRACT. A space X is called s-point finite refinable (ds-point finite refinable) provided every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ such that, for some (closed discrete) $C \subseteq X$,

(i) for all nonempty $V \in \mathcal{V}$, $V \cap C \neq \emptyset$ and
(ii) for all $a \in C$ the set $(\mathcal{V})_a = \{V \in \mathcal{V} : a \in V\}$ is finite.

In this paper we distinguish these spaces, study their basic properties and raise several interesting questions. If λ is an ordinal with $cf(\lambda) = \lambda > \omega$ and S is a stationary subset of λ then S is not s-point finite refinable. Countably compact ds-point finite refinable spaces are compact. A space X is irreducible of order ω if and only if it is ds-point finite refinable. If X is a strongly collectionwise Hausdorff ds-point finite refinable space without isolated points then X is irreducible.

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Suppose that $\mathcal{U}$ is a cover of a set X and $C \subseteq X$. We say that $\mathcal{U}$ is *C-point finite* provided

(i) for all nonempty $U \in \mathcal{U}$, $U \cap C \neq \emptyset$,
(ii) for all $a \in C$ the set $(\mathcal{U})_a = \{U \in \mathcal{U} : a \in U\}$ is finite.

We will call a space X *s-point finite refinable (ds-point finite refinable)* provided every open cover $\mathcal{U}$ of X has a $C_{\mathcal{U}}$-point finite open refinement $\mathcal{V}$ for some (closed discrete) $C_{\mathcal{U}} \subseteq X$. Notice that if $\mathcal{U}$ is a C-point finite cover of a space X for some $C \subseteq X$ and $\emptyset \notin \mathcal{U}$ then condition (i) above can be restated as $\mathcal{U} = \cup\{(\mathcal{U})_a : a \in C\}$. In [10] Chaber observed that if $\mathcal{U}$ is an open cover of a countably compact space X then $\{st(x,\mathcal{U}) : x \in X\}$ has a finite subcover. That is, there is a finite $C \subseteq X$ such that the collection $\cup\{(\mathcal{U})_a : a \in C\}$ is a subcover of $\mathcal{U}$. A T_1-space X is compact (a T_3-space Lindelöf) if and only if every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ which is C-point finite for some finite (countable) $C \subseteq X$. More generally, in [2, 1] it is noted that if $\mathcal{U}$ is an open cover of a T_1-space X then there is a closed discrete $A \subseteq X$ such that $\mathcal{U}_A = \cup\{(\mathcal{U})_a : a \in A\}$ is a subcover of $\mathcal{U}$. It is readily seen that a (T_3-) space X is metacompact (paracompact) if and only if for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V}$ such that for some closed discrete set $C \subseteq X$, $\mathcal{V}$ is C-point finite and

(iii) the collection $\{st(a,\mathcal{V}) : a \in C\}$ is point finite (locally finite).

A collection of subsets $\mathcal{H}$ of a set X is called *minimal* provided that for all $H \in \mathcal{H}$, $\cup(\mathcal{H}\backslash\{H\}) \neq \cup\mathcal{H}$. A topological space is called *irreducible* provided every open cover has a minimal open refinement. This concept was introduced in [11] where it is

shown that every point finite cover of a set has a minimal subcover. It follows from [6, Thm. 1.1], that a space X is irreducible if and only if every open cover $\mathcal{U}$ has a C-point finite open refinement $\mathcal{V}$ for some closed discrete set $C \subseteq X$ such that for all $a \in C$, $|\mathcal{V}_a| = 1$. We show that every space X can be embedded as a closed subset of an irreducible space, Theorem 1.3. We do not know if every ds-point finite refinable space is irreducible. However countably compact ds-point finite refinable spaces are compact and we show that ds-point finite refinable spaces are irreducible of order ω, Theorem 2.5. Also strongly collectionwise Hausdorff ds-point finite refinable spaces without isolated points are irreducible, Corollary 2.3.

Many covering properties have been characterized in terms of monotone or directed open covers. For example, a space is metacompact if and only if every monotone open cover has a point finite open refinement, [17]. For ds-point finite refinability we cannot restrict ourselves to monotone open covers, Example 3. However the question for directed open covers remains open. We have several results that demonstrate the central importance of this question. Suppose that $\mathcal{U}$ is a directed open cover of a T_1-space X and that C is a closed discrete subset of X. If $\mathcal{U}$ has an open C-point finite refinement then it has a minimal open C-point finite refinement, Theorem 2.1. Suppose $f : X \to Y$ is a perfect map from a space X onto a ds-point finite refinable space Y. Every directed open cover of X has a minimal open refinement, Theorem 3.2.

Throughout this paper all spaces are T_1. When we topologize an ordinal space it will be with the order topology and subsets of topological spaces will have the subspace topology. If X is a set, $\mathcal{H}$ a collection of subsets of X and $C \subseteq X$ then $\mathcal{H}$ is said to be *point finite on* C provided $C \subseteq \cup\mathcal{H}$ and every $x \in C$ is in only finitely many members of $\mathcal{H}$. For any collection $\mathcal{A}$ of subsets of a set X and any $x \in X$, $(\mathcal{A})_x = \{A \in \mathcal{A} : x \in A\}$ and $st(x, \mathcal{A}) = \cup(\mathcal{A}_x)$.

1. Examples and basics

Theorem 1.1. *If λ is an ordinal with $cf(\lambda) = \lambda > \omega$ and S is a stationary subset of λ then S is not s-point finite refinable.*

Proof. Suppose that $\mathcal{V}$ is an open refinement of the open cover $\{[0, \alpha] \cap S : \alpha \in S\}$ of S. For every limit ordinal $\alpha \in S$ let $V_\alpha \in (\mathcal{V})_\alpha$ and $\beta(\alpha) < \alpha$ such that $[\beta(\alpha), \alpha] \cap S \subseteq V_\alpha$. By the "Pressing Down Lemma" there is a $\beta^* < \lambda$ such that the set $S' = \{\alpha \in S : \alpha$ is a limit ordinal and $\beta^* = \beta(\alpha)\}$ is a stationary subset of λ. Notice that if $\alpha \in S$ and $(\mathcal{V})_a$ is finite then $\alpha < \beta^*$. Thus if $D \subseteq S$ and $\mathcal{V}$ is point finite on D then $D \subseteq [0, \beta^*]$ and therefore $|D| < \lambda$.

Suppose $D \subseteq S$ and $\mathcal{V}$ is D-point finite. Then, since $|D| < \lambda$ and $|S'| = \lambda$, there is an $\eta \in D$ such that the set $S'' = \{\alpha \in S' : \eta \in V_a\}$ has cardinality λ. Since $(\mathcal{V})_\eta$ is finite there is a $V \in \mathcal{V}$ such that $S^* = \{\alpha \in S'' : V = V_\alpha\}$ has cardinality λ. But this is not possible since $V \subseteq [0, \gamma]$ some $\gamma < \lambda$ and therefore has cardinality less than λ. Hence $\mathcal{V}$ is not D-point finite for any $D \subseteq S$. □

The following clearly holds.

Theorem 1.2. *Every ds-point finite refinable, countably compact (T_3 $\aleph_1$-compact) space is compact (Lindelöf).*

Example 1. A countably compact, noncompact s-point finite refinable LOTS which is not ds-point finite refinable.

Let $X = \omega_1 \times (\omega_1 + 1)$ be given the lexicographic order. That is, define a linear ordering $<^*$on X by letting $(\alpha,\beta) <^* (\gamma,\delta)$ provided

$$\alpha < \gamma, \quad \text{or} \quad \alpha = \gamma \quad \text{and} \quad \beta < \delta \quad \text{for all } \alpha,\gamma < \omega_1 \text{ and } \beta,\delta \le \omega_1. \tag{1}$$

We topologize X by giving it the order topology. Since X is countably compact but not compact it is not ds-point finite refinable.

Notice that for a limit ordinal $\beta < \omega_1$, if $x \in X$ and $x <^* (\beta,0)$, then there is an ordinal $\alpha < \beta$ such that $x <^* (\alpha,0)$. For ordinals $\alpha < \beta < \omega_1$, define

$$\begin{aligned}(\alpha,\beta]^* &= \{x \in X : (\alpha,0) <^* x \le^* (\beta,0)\} \\ &= \cup\{\{(\gamma,\delta) : \delta \le \omega_1\} : \alpha \le \gamma < \beta\} \setminus \{(\alpha,0)\} \cup \{(\beta,0)\}.\end{aligned} \tag{2}$$

If β is a limit ordinal such sets form an open local base at the point $(\beta,0)$. Clearly, $\{(\alpha,0) : \alpha < \omega_1\}$ is a closed subset of X homeomorphic to ω_1 with the usual order topology.

Claim. *X is s-point finite refinable.*

Proof. Let $\mathcal{V}$ be an open cover of X. For each limit ordinal $\beta < \omega_1$, let $V_\beta \in \mathcal{V}$ such that $(\beta,0) \in V_\beta$ and let $\alpha(\beta) < \beta$ such that $(\alpha(\beta),\beta]^* \subseteq V_\beta$. By the Pressing Down Lemma, there is an uncountable set $B \subseteq \omega_1$ and an ordinal $\gamma < \omega_1$ such that $\gamma = \alpha(\beta)$ for all $\beta \in B$. For all $\beta \in B$ let $U_\beta = \{(\gamma,\beta+1)\} \cup (\gamma+1,\beta]^*$ and let $\mathcal{U} = \{U_\beta : \beta \in B\}$. Note that

(1) $\{x \in X : (\gamma+1,0) <^* x\} \subseteq \cup\mathcal{U}$.

(2) for all $\beta \in B$, U_β is the only member of $\mathcal{U}$ containing the isolated point $(\gamma,\beta+1)$.

The set $[0,\gamma+1]^*$ is compact, since $[0,\gamma+1]^*$ is a closed subset of the compact subspace $[0,\gamma+1] \times (\omega_1+1)$ of X. Thus, there is a finite subset $\mathcal{W} = \{W_1,W_2,\dots,W_n\}$ of $\mathcal{V}$ that covers $[0,\gamma+1]^*$. For each $k \in \{1,2,\dots,n\}$ let $x_k \in W_k$. Note that if $k \in \{1,2,\dots,n\}$ and $\beta \in B$ with $x_k \in U_\beta$ then $x_k = (\gamma,\beta+1)$. Thus, x_k is in at most 1 member of $\mathcal{U}$ for each $k \in \{1,2,\dots,n\}$. Let $C = \{(\gamma,\beta+1) : \beta \in B\} \cup \{x_1,x_2,\dots,x_n\}$. The collection $\mathcal{W} \cup \mathcal{U}$ is an open refinement of $\mathcal{V}$ and

(1) $G \cap C \ne \varnothing$ for all $G \in \mathcal{W} \cup \mathcal{U}$,

(2) each member of C is in at most $n+1$ members of $\mathcal{W} \cup \mathcal{U}$.

Hence $\mathcal{W} \cup \mathcal{U}$ is a C-point finite open refinement of $\mathcal{V}$. Thus, X is s-point finite refinable. □

Metacompact spaces (among many others) are isocompact. That is every closed countably compact subset of a metacompact space is compact [3].

Example 2. Irreducible GO-space Y which is not isocompact.

Let X be as in Example 1 and $Y = (\omega_1 \times \{0\}) \cup \{(\alpha,\beta+1) : \beta < \omega_1\}$ be given the subspace topology. Clearly, $\{(\alpha,0) : \alpha < \omega_1\}$ is a closed subset of Y homeomorphic to ω_1 with the usual order topology and all other points of Y are isolated. Hence, Y is not isocompact. In a manner similar to the proof that Example 1 is s-point finite refinable we can show that Y is irreducible.

Closed subspaces of s-point finite refinable, ds-point finite refinable and even irreducible spaces need not be s-point finite refinable. In fact, Examples 1 and 2 both have a closed subset homeomorphic to ω_1. We can say more.

Theorem 1.3. *Every space X can be embedded as a closed subspace of an irreducible space Y.*

Proof. Let Y be the space obtained from the product space $X\times(\omega+1)$ by isolating the points of $X\times\omega$. Suppose $\mathcal{U}$ is an open cover of Y. For each $x\in X$, choose U_x open in X containing x and $n_x\in\omega$ such that there is a $U\in\mathcal{U}$ with $U_x\times[n_x,\omega]\subseteq U$. Let $X_m=\{x\in X: n_x=m\}$ for all $m\in\omega$. Let $\mathcal{U}_0=\{(U_x\times[1,\omega]\cup\{(x,0)\}: x\in X_0\}$. For $n>0$, let $\mathcal{U}_n=\{(U_x\times[n+1,\omega])\cup\{(x,n)\}: x\in X_n$ and $x\notin\cup_{k<n}\cup_{y\in X_k}U_y\}$. Finally, let $\mathcal{U}_\omega=\{\{x\}: x\in Y$ and $x\notin\cup\cup_{n<\omega}\mathcal{U}_n\}$. Let $\mathcal{V}=\cup_{n\le\omega}\mathcal{U}_n$. Clearly, $\mathcal{V}$ is an open refinement of $\mathcal{U}$. Suppose $V\in\mathcal{V}$. If $V\in\mathcal{U}_\omega$, then $V=\{x\}$ and x is not in any other element of $\mathcal{V}$. If $V\in\mathcal{U}_n$, for $n<\omega$, then choose $x\in X$ such that $V=(U_x\times[n+1,\omega])\cup\{(x,n)\}$. We show that $(x,n)\notin W$ for any $W\in\mathcal{V}\setminus\{V\}$. If $W\in\mathcal{U}_\omega,(x,n)\notin W$ since $(x,n)\in V$. If $W\in\mathcal{U}_k$ for some k, then either $k=n$, $k>n$, or $k<n$. If $k<n$, then $(x,n)\notin W$ by the definition of $\mathcal{U}_n$. If $k>n$, then $W\subseteq X\times[n+1,\omega]$ so $(x,n)\notin W$. If $k=n$, and $(x,n)\in W$, then $W=V$. Hence $\mathcal{V}$ is minimal. Thus Y is an irreducible space, and clearly X is homeomorphic to the closed subspace Y^o. □

Every space contains a left-separated dense subspace [16, 2.6(ii)]. M. Ismail has observed that in [14, Lem. 2.2] where it is shown that every left-separated T_2-space is C-closed (countably compact subsets are closed), they in fact prove that such spaces are hereditarily irreducible.

Theorem 1.4 (Ismail)**.** *Every T_2-space X has a dense hereditarily irreducible subspace.*

In [13, Thm. 2.3] they characterize paracompact GO-spaces as those GO-spaces which do not have a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal. GO-spaces are monotonically normal and in [4] they show that this elegant characterization of paracompactness in GO-spaces remarkably holds for the broader class of monotonically normal spaces.

Theorem 1.5 (Balogh and Rudin)**.** *A monotonically normal space is paracompact if and only if it does not have a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.*

Example 1 shows that s-point finite refinable LOTS and Example 2 irreducible GO-spaces can have closed subsets homeomorphic to ω_1 and therefore, in spite of Theorem 1.1, need not be paracompact. However, H. Bennett has proved the following surprising result.

Theorem 1.6 (Bennett)**.** *A LOTS is paracompact if and only if it is ds-point finite refinable.*

2. Are ds-point finite refinable spaces irreducible? Clearly, irreducible spaces are ds-point finite refinable and as shown in Example 1, s-point finite refinable spaces

need not be irreducible.

Question 1. Is every ds-point finite refinable space irreducible?

The following is readily verified.

Theorem 2.1. *A space X is irreducible, if X^o, the set of all nonisolated points of X, is irreducible.*

Notice that X being irreducible does not necessarily imply that the subspace X^o is irreducible or even s-point finite refinable, see Example 1.4. In fact, Theorem 1.3, shows that any space X can be embedded as a closed subspace of an irreducible space Y with $X = Y^o$.

A space X is said to be *(strongly) collectionwise Hausdorff* if for any closed discrete set A in X, the points of A can be separated by a (discrete) disjoint collection of open sets.

Theorem 2.2. *Suppose that X is a strongly collectionwise Hausdorff space. If X^o is ds-point finite refinable then X is irreducible.*

Proof. Suppose that $\mathcal{U}$ is an open cover of X. Let $\mathcal{U}^o = \{U \cap X^o : U \in \mathcal{U}\}$ and $\mathcal{V}^o$ be an open (in X^o) D-point finite refinement of $\mathcal{U}^o$ where D is a closed discrete subset of X^o. For all $\emptyset \neq V \in \mathcal{V}^o$ let $W(V)$ be an open subset of X such that $V = W(V) \cap X^o$ and $W(V) \subseteq U$ for some $U \in \mathcal{U}$. Notice that $\mathcal{W} = \{W(V) : V \in \mathcal{V}^o\}$ is an open partial refinement of $\mathcal{U}$ covering X^o such that

(1) $W \cap D \neq \emptyset$, for all $W \in \mathcal{W}$,

(2) for all $x \in D$, $(\mathcal{W})_x$ is finite.

Since D is closed discrete and X^o which is closed in X, D is closed discrete in X. Since X is strongly CWH, for each $x \in D$, let $G(x)$ be an open neighborhood of x such that $\mathcal{G} = \{G(x) : x \in D\}$ is discrete. For all $x \in D$, let $H(x) = G(x) \cap (\cap (W)_x)$ and notice, by (2), $H(x)$ is open. Since $\mathcal{G}$ is discrete so is $\mathcal{H} = \{H(x) : x \in D\}$. Also, since no point of D is isolated in X, for all $x \in D$, $H(x) \backslash \{x\}$ is infinite. For each $x \in D$, let $n_x = |(W)_x| < \omega$ and for $i = 1, 2, \ldots, n_x$, let $y_i^x \in H(x) \backslash \{x\}$ such that if $i, k \in \{1, 2, \ldots, n_x\}$ with $i \neq k$ then $y_i^x \neq y_k^x$. Since $\mathcal{H}$ is discrete so is $D^* = \{y_i^x : x \in D$ and $i \in \{1, 2, \ldots, n_x\}\}$. For each $x \in D$, list $(\mathcal{W})_x = \{V_1^x, \ldots, V_{n_x}^x\}$ and, for each $i \in \{1, \ldots, n_x\}$, let $W_i^x = V_i^x \backslash (D^* \backslash \{y_i^x\})$. The collection $\mathcal{W}^* = \{W_i^x : x \in D, i \in \{1, 2, \ldots, n_x\}\}$ is clearly a minimal open partial refinement of $\mathcal{W}$ and hence $\mathcal{U}$ covering X^o. Then $\mathcal{V} = \mathcal{W}^* \cup \{\{x\} : x \in X \backslash \cup \mathcal{W}^*\}$ is a minimal open refinement of $\mathcal{U}$. □

Corollary 2.3. *If X is a strongly collectionwise Hausdorff ds-point finite refinable space without isolated points then X is irreducible.*

Clearly normal collectionwise Hausdorff spaces are strongly collectionwise Hausdorff. A T_2-space is said to be of *point-countable type* if every point is contained in a compact set of countable character. For example, first countable spaces, locally compact space, and even p-spaces [12]. $V = L$ implies that normal spaces of point-countable type are collectionwise Hausdorff [19].

Corollary 2.4. *$V = L$ implies that if X is a normal space of point-countable type such that X^o is ds-point finite refinable then X is irreducible.*

A collection $\mathcal{S}$ of sets is said to be *monotone* provided for all $S, S' \in \mathcal{S}$ either $S \subseteq S'$ or $S' \subseteq S$. More generally, collection $\mathcal{S}$ of sets is said to be *directed* provided for all S, $S' \in \mathcal{S}$ there is a $T \in \mathcal{S}$ such that $S \cup S' \subseteq T$. A space X is metacompact if and only if every monotone open cover of X has a point-finite open refinement [18].

Example 3. A non ds-point finite refinable space in which every monotone open cover has a minimal open refinement.

Let $Y = \prod_{i=1}^{\infty}(\omega_i + 1)$. For each natural number k, let $X_k = (\prod_{i=1}^{k}(\omega_i + 1)) \times (\prod_{i=k+1}^{\infty}(\omega_i))$ and let $X = \cup_{k=1}^{\infty} X_k$ with the usual subspace topology. The space X is $\aleph_1$-compact and every monotone open cover of X has a countable subcover but X is not Lindelöf, [19]. Since a countable open cover of any space has an irreducible open refinement, every monotone open cover of X has an irreducible open refinement. However, since X is $\aleph_1$-compact but not Lindelöf, X is not ds-point finite refinable.

Notice that if $\mathcal{V}$ is an open cover of Example 3 having no countable subcover and we let $\mathcal{V}^*$ be the collection of all finite unions of members of $\mathcal{V}$ then $\mathcal{V}^*$ is a directed open cover of X which does not have a C-point finite open refinement for any closed discrete $C \subseteq X$.

Theorem 2.5. *Suppose that $\mathcal{U}$ is a directed open cover of a T_1-space X and that C is a closed discrete subset of X. If $\mathcal{U}$ has an open C-point finite refinement then it has a minimal open C-point finite refinement.*

Proof. Suppose that $\mathcal{V}$ is an open C-point finite refinement of $\mathcal{U}$. For each $V \in \mathcal{V}$ and each $x \in C \cap V$, let $W(V,x)$. Since the set C is closed discrete for each $x \in C$ the set $C \setminus \{x\}$ is closed. Hence for each $V \in \mathcal{V}$ and each $x \in C \cap V$ the set $W(V,x)$ is open. Let $\mathcal{W} = \{W(V,x) : V \in \mathcal{V} \text{ and } x \in C \cap V\}$ and $\mathcal{W}' = \{st(x,\mathcal{W}) : x \in C\}$. Since $\mathcal{V}$ is an open cover of X and for each $V \in \mathcal{V}$ and each $x \in C \cap V$, $W(V,x) = (V \setminus C) \cup \{x\} \subseteq V$ the collection $\mathcal{W}$ is an open refinement of $\mathcal{U}$. Since $\mathcal{U}$ is directed and elements of $\mathcal{W}'$ are finite unions of elements of $\mathcal{W}$, $\mathcal{W}'$ is an open refinement of $\mathcal{U}$. To see that $\mathcal{W}'$ is minimal we need only note that for every $x \in C$ the set $st(x,\mathcal{W})$ is the only member of $\mathcal{W}'$ containing x. □

Question 2. If every directed open cover $\mathcal{U}$ of a space X has a $C_{\mathcal{U}}$-point finite open refinement for some closed discrete $C_{\mathcal{U}} \subseteq X$, is X irreducible?

It follows from Theorem 2.5 that the answer to Question 2 being yes for irreducible spaces would imply an affirmative answer to Question 1.

In [7] the concept of irreducible of order κ is introduced for any infinite cardinal κ, generalizing irreducible. In this paper, we are interested only in the case where $\kappa = \omega$. A space X is *irreducible of order* ω provided for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n$ and a family of discrete closed collections $\{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n\}$ such that

(1) For all $k \in \{1,2,\ldots,n\}$ and for all $T \in \Upsilon_k, \mathcal{V}_T^k = \{V \in \mathcal{V}_k : T \subseteq V\} \neq \emptyset$ but finite.

(2) $\{V : V \in \mathcal{V}_T^k, T \in \Upsilon_k, k \in \{1,2,\ldots,n\}\}$ covers X.

Theorem 2.6. *A space X is irreducible of order ω if and only if it is ds-point finite refinable.*

Proof. ($\Leftarrow$) Clearly ds-point finite refinable implies irreducible of order ω.

($\Rightarrow$) Suppose X is irreducible of order ω and $\mathcal{U}$ is an open cover of X. Let $\mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n$ be an open refinement of $\mathcal{U}$ and suppose $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n$ are discrete closed collections such that

(1) For all $k \in \{1,2,\ldots,n\}$ and for all $T \in \Upsilon_k, \mathcal{V}_T^k = \{V \in \mathcal{V}_k : T \subseteq V\} \neq \emptyset$ but finite.

(2) $\{V : V \in \mathcal{V}_T^k, T \in \Upsilon_k,\ k \in \{1,2,\ldots,n\}\}$ covers X.

For each $k \in \{1,2,\ldots,n\}$ and $T \in \Upsilon_k$ let $x(k,T) \in T$ and note that $A_k = \{x(k,T) : T \in \Upsilon_k\}$ is a discrete collection since Υ_k is discrete and $A = A_1 \cup \cdots \cup A_n$ is also closed discrete. For each $k \in \{1,2,\ldots,n\}$, $T \in \Upsilon_k$ and $V \in \mathcal{V}_T^k$ let $G(k,T,V) = V \setminus (A \setminus \{x(k,T)\})$ and $\mathcal{G} = \{G(k,T,V) : k \in \{1,2,\ldots,n\}, T \in \Upsilon_k, V \in \mathcal{V}_T^k\}$. Since A is closed discrete, $\mathcal{G}$ is a collection of open sets. Let $y \in A$. There is a $k \in \{1,2,\ldots,n\}$ and a $T \in \Upsilon_k$ such that $y = x(k,T) \in T$. By (1), $\mathcal{V}_T^k \neq \emptyset$ so let $V \in \mathcal{V}_T^k$ and $y \in G(k,T,V)$.

Suppose $y \in X \setminus A$. By (2) there is a $k \in \{1,2,\ldots,n\}$, $T \in \Upsilon_k$ and $V \in \mathcal{V}_T^k$ such that $y \in V$. Since $G(k,T,V) \setminus A = V \setminus A$, $y \in G(k,T,V)$. Hence $\mathcal{G}$ is an open cover of X. For $k \in \{1,2,\ldots,n\}$, $T \in \Upsilon_k$ and $V \in \mathcal{V}_T^k$, $G(k,T,V) \subseteq V$. Hence, since $\mathcal{V}$ refines $\mathcal{U}$, $\mathcal{G}$ refines $\mathcal{U}$.

Finally suppose $y \in A$. If $k \in \{1,2,\ldots,n\}$, $T \in \Upsilon_k$ and $V \in \mathcal{V}_T^k$ such that $y \in G(k,T,V)$ then $y = x(k,T)$. Since for each $k \in \{1,2,\ldots,n\}$ there is at most one $T \in \Upsilon_k$ such that $y = x(x,T)$ (Υ_k discrete (pwd)) and for each $T \in \Upsilon_k$ $\mathcal{V}_T^k$ is finite, the point y can be in only finitely many members of $\mathcal{G}$. Thus $\mathcal{G}$ is an A-point finite refinement of $\mathcal{U}$. Since A is closed discrete, we conclude that X is ds-point finite refinable. □

Question 1 can be restated as "Are irreducible and irreducible of order ω equivalent?".

3. Mappings. The properties of the image and pre-image of spaces having various covering properties under a variety of mappings have been well studied (see [8, 9]). In particular, the perfect pre-image of a metacompact space is metacompact [12] and the closed image of a metacompact space is metacompact [18]. For s-point finite refinable spaces little positive is known. In the next two theorems the possible significance of the question of characterizing these spaces in terms of directed open covers (Question 2) is seen. Notice that the proof of Theorem 3.1 is a modification of the proof of [12, Thm. 5.1.35].

Theorem 3.1. *Suppose $f : X \to Y$ is a perfect map from a space X onto an s-point finite refinable space Y. Every directed open cover of X has a C-point finite open refinement for some set $C \subseteq X$.*

Proof. Let $\mathcal{U}$ be a directed open cover of X. For each $y \in Y$, let $U(y) \in \mathcal{U}$ such that $f^{-1}(y) \subseteq U(y)$ ($f^{-1}(y)$ is compact). For each $y \in Y$, let $V(y)$ be an open neighborhood of y in Y such that $f^{-1}(y) \subseteq f^{-1}(V(y)) \subseteq U(y)$ (f is closed, continuous). Now $\mathcal{V} = \{V(y) : y \in Y\}$ is an open cover of Y so let $A \subseteq Y$ and $\mathcal{W}$ an open refinement of $\mathcal{V}$ such that

(1) $W \cap A \neq \emptyset$ for all $W \in \mathcal{W}$.

(2) $\{W \in \mathcal{W} : a \in W\}$ is finite for all $a \in A$.

For each $a \in A$ let $x_a \in f^{-1}(a)$ and $A^* = \{x_a : a \in A\}$. The collection $\{f^{-1}(W) : W \in \mathcal{W}\}$ is an A^*-point finite refinement of $\mathcal{U}$. Let $W \in \mathcal{W}$, and since $W \cap A \neq \emptyset$, let $a \in W \cap A$ and note that $x_a \in f^{-1}(a) \subseteq f^{-1}(W)$. Thus $f^{-1}(W) \cap A^* \neq \emptyset$. Now

let $a \in A$ and suppose $x_a \in f^{-1}(W)$. Then $a = f(x_a) \in W$. However, the collection $\{W \in \mathcal{W} : a \in W\}$ is finite so $\{f^{-1}(W) : W \in \mathcal{W}$ and $x_a \in f^{-1}(W)\}$ is also finite. □

Notice in the above proof if the set A is closed discrete then the set A^* is also closed discrete. The following result is a direct consequence of this and Theorem 2.5.

Theorem 3.2. *Suppose $f : X \to Y$ is a perfect map from a space X onto a ds-point finite refinable space Y. Every directed open cover of X has a minimal open refinement.*

The following examples show that even irreducible spaces are not very well behaved under closed mappings.

Example 4. The perfect image of an s-point finite refinable space need not be s-point finite refinable.

Let X be the space from Example 1. Define a function $f : X \to \omega$ as follows

$$f(\alpha,\beta) = \begin{cases} \alpha, & \text{if } \beta = 0 \\ \alpha+1, & \text{otherwise} \end{cases} \quad \text{for all } \alpha < \omega_1 \text{ and } \beta \le \omega_1. \tag{3}$$

Example 5. The closed image of an irreducible space need not be s-point finite refinable.

The restriction of the function in Example 4 to the subspace Y of X in Example 2 is a closed mapping onto ω_1.

Question 3. Is the perfect pre-image of an s-point finite refinable (ds-point finite refinable) [irreducible] space s-point finite refinable (ds-point finite refinable) [irreducible]?

Notice that if the answer to Question 1 is yes for irreducible spaces then Theorem 3.2 implies that the perfect pre-image of an irreducible space is irreducible.

Question 4. Is the perfect image of a ds-point finite refinable (irreducible) space ds-point finite refinable (irreducible)?

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Davis, Grabner, and Grabner: 229 Vincent Science Hall, Slippery Rock, PA 16057-1326, USA

E-mail address: gary.grabner@sru.edu