ALMOST TRIANGULAR MATRICES OVER DEDEKIND DOMAINS

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Abstract. Every matrix over a Dedekind domain is equivalent to a direct sum of matrices $A = (a_{i,j})$, where $a_{i,j} = 0$ whenever $j > i + 1$.

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1. Introduction. Two $m \times n$ matrices $A$ and $B$ over a ring $R$ are called equivalent if $B = PAQ$ for invertible matrices $P$ and $Q$ over $R$. From now on, assume that $R$ denotes a Dedekind domain with quotient field $K$. If $I = (a, b)$ is a non principal ideal in $R$, then, in contrast with the situation for Principal Ideal Domains, the $1 \times 2$ matrix $[a, b]$ is not equivalent over $R$ to a matrix whose off diagonal entries are 0. Using the separated divisor theorem in the form given by Levy in [2], other facts about matrices over Dedekind domains in [2], and elementary properties of ideals in Dedekind domain [1], we show that any $m \times n$ matrix over a Dedekind domain is equivalent to a direct sum of matrices $A = (a_{i,j})$ with $a_{i,j} = 0$ when $j > i + 1$. If the direct summand $A$ has rank $r$, then the number of rows, respectively columns, of $A$ is either $r$ or $r + 1$. The corresponding result for similarity of matrices over principal ideal rings is that every $n \times n$ matrix over a principal ideal ring is similar to an upper triangular matrix [3, p. 42].

2. Diagonalization of matrices. If $A$ is an $m \times n$ matrix, then $A$ can be viewed as an $R$-module homomorphism $A : R^n \to R^m$ by left multiplication. If $M_A$ denotes the submodule of $R^m$ generated by the columns of $A$, then $M_A$ is the image of $A$ in $R^m$ and the isomorphism class of the cokernel $S_A = R^m/M_A$ of $A$ determines the equivalence class of $A$.

Separated divisor theorem [2]. There is a chain of integral $R$-ideals $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_r$ and a fractional $R$-ideal $H$ such that

$$S_A = \begin{cases} \oplus_{i=1}^r L_i \oplus H \oplus R^{m-r-1}, & m < r \\ \oplus_{i=1}^r E_i, & m = r, \end{cases}$$

where $H = \prod_{i=1}^r L_i$ if $r = n$ and $H \supseteq R$ if $r = 0$ or $r = m$.

The isomorphism class of $S_A$, the ideals $\{L_i\}_{i=1}^r$ (as sets), and the isomorphism class of $H$ both determine and are determined by the equivalence class of $A$. 

We also need the following elementary facts about ideals in Dedekind domains.

**Lemma 1** [1, p. 150, 154]. Let I, J be integral ideals in R. Then

1. There is an $\alpha$ in the quotient field $K$ of $R$ such that $\alpha I$ is integral and $\alpha I + J = R$;
2. There is an $R$-module isomorphism $\gamma : IJ \oplus R \rightarrow I \oplus J$, given by $\gamma(u, v) = (x_1v - u, \alpha u - x_2v)$, where $\alpha$ is as in (1) and $x_1 \in I$, $x_2 \in J$ are chosen with $\alpha x_1 - x_2 = 1$.

**Note.** The $R$-linear homomorphism $\gamma$ is given by the matrix $\begin{pmatrix} -1 & x_1 \\ \alpha & -x_2 \end{pmatrix}$, where $\alpha \in K$.

**Theorem 2.2.** Every $m \times n$ matrix $A$ over a Dedekind domain is equivalent to a direct sum of matrices $(a_{ij})$ with $a_{ij} = 0$ whenever $j > i + 1$.

**Proof.** An $m \times n$ matrix $A$ is called indecomposable if $A$ is not equivalent to a matrix of the form $\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$ for any matrices $B_1$, $B_2$. That is, $A$ is not equivalent to a direct sum of matrices $B_1$, $B_2$. If $A = 0$, the result is clear. Assume that $A \neq 0$. It is sufficient to verify the result for indecomposable matrices. In this case, if $r$ is the rank of $A$ over the quotient field $K$ of $R$, then [2, Lem. 2.1] asserts that $m = r$ or $r + 1$ and $n = r$ or $r + 1$. There are then four possible cases to check.

**Case 1.** Assume that $m = r$ and $n = r$. Then $S_A = \oplus_{i=1}^{r} R/L_i$, with $L_1, \ldots, L_r$ integral $R$-ideals with $L_1 \leq L_2 \leq \cdots \leq L_r$ and $\prod_{i=1}^{r} L_i = \alpha x$. Thus, $\prod_{i=1}^{r} L_i = \langle \alpha \rangle$ is a principal ideal generated by $\alpha \in R$. Let $\phi_0 : R^r \rightarrow \prod_{i=1}^{r} L_i \oplus R^{r-1}$ be given by $\phi_0(r_1, \ldots, r_r) = (ar_1, r_2, \ldots, r_r)$ and let $\phi_1 : L_1 \oplus \cdots \oplus L_{r-1} \oplus R \oplus R^{r-1-1} \rightarrow L_1 \oplus \cdots \oplus L_r \oplus \prod_{i=j+1}^{r} L_i \oplus R^{r-1-1}$ be given by $\phi_j = I_{r-j} \oplus y_j \oplus I_{r-j-1}$, where $y_j : \prod_{i=j+1}^{r} L_i \oplus R \rightarrow L_j \oplus \prod_{i=j+1}^{r} L_i$ is the map given in Lemma 1 and $I_{r-j-1}, I_{r-j-1}$ are the identity maps of indicated rank. Let $\phi : R^r \rightarrow L_1 \oplus \cdots \oplus L_r \subset R^r$ be given by $\phi = \phi_{r-1} \phi_{r-2} \cdots \phi_1 \phi_0$. Then the matrix $[\phi]$ of $\phi$, with respect to the standard bases for $R^r$, is: $[\phi] = [\phi_{r-1}] \cdots [\phi_1] [\phi_0]$.

While $[\phi_1]$ may have entries which are not in $R$, $[\phi]$ has all its entries in $R$ since each $L_j$ is integral. If we write

$$[\phi] = \begin{pmatrix} I_{r-j} & 0 & 0 & 0 \\ 0 & -1 & x_j^1 & 0 \\ 0 & \alpha_j & -x_j^2 & 0 \\ 0 & 0 & 0 & I_{r-j-1} \end{pmatrix},$$

then a direct calculation shows that

$$[\phi] = \begin{pmatrix} -a & x_1^1 & 0 & 0 & 0 & 0 & 0 \\ -a \alpha_1 & -x_2^1 & x_1^2 & 0 & 0 & 0 & 0 \\ -a \alpha_1 \alpha_2 & \alpha_2 x_1^1 & x_2^2 & x_1^3 & 0 & 0 & 0 \\ -a \alpha_1 \alpha_2 x_2 & \alpha_2 \alpha_3 x_1^2 & x_3^2 & x_1^4 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a \prod_{i=1}^{r-1} \alpha_i & \cdots & \cdots & \cdots & \cdots & \alpha_{r-2} x_2^{r-2} & x_2^{r-1} \end{pmatrix}.$$

Since $[\phi]$ has the same number of rows and columns and the same cokernel as $A$, $[\phi]$ is equivalent to $A$. 

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Remark. Assume that $L_i = \langle a_i \rangle$ is principal for each $i, i = 1, \ldots, r$ and $a_i \in R$. The isomorphism $y_j : \prod_{i=j}^r L_i \oplus R \cong L_j \oplus \prod_{i=j+1}^r L_i$ can be given as $y_j(u, v) = (\alpha_j u, \beta_j v)$, where $\alpha_j = 1 / \prod_{i=j+1}^r a_i$ and $\beta_j = \prod_{i=j+1}^r a_i$. In this case, $[\phi] = \text{diag}[a_1, \ldots, a_r]$ with $a_i | a_{i+1}$ for $1 \leq i \leq r$. This is the only case which occurs if $R$ is a PID.

Case 2. Assume that $m = r$ and $n = r + 1$. Then $S_A = \oplus_{i=1}^r R/L_i$ with $L_i, 1 \leq i \leq r$ integral ideals and $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_r$. Let $L_{r+1}$ be integral ideal with $\prod_{i=1}^{r+1} L_i = \langle a \rangle$ principal, then $\oplus_{i=1}^{r+1} L_i \cong R^n$ and there is a chain of $R$-homomorphisms

$$R^n \xrightarrow{\phi} L_1 \oplus \cdots \oplus L_r \oplus L_{r+1} \xrightarrow{\pi} L_1 \oplus \cdots \oplus L_r \subseteq R^r,$$

(2.4)

where $\pi$ is the projection on $L_1 \oplus \cdots \oplus L_r$ along $L_{r+1}$. The matrix of $\pi \circ \phi$ is an $m \times n$ matrix obtained by deleting the last row of $[\phi]$ and, thus, has the same form as in Case 1. Since the cokernel of $\pi \circ \phi$ is the same as $A$ and $[\pi \circ \phi]$ has the same number of rows and columns as $A, [\pi \circ \phi]$ is equivalent to $A$.

Case 3. Assume that $m = r + 1$ and $n = r$. Then $S_A = \oplus_{i=1}^r R/L_i \oplus H$, where $L_i, 1 \leq i \leq r$ are integral ideals and $H \cong \prod_{i=1}^r L_i$. Choose $a \in R$ with $L_i H^{-1}a$ integral. Note that $L_i H^{-1}a$ is a submodule of $H^{-1}a$. From Case 1, we construct an $R$-isomorphism $\phi_r : R^r \rightarrow L_1 \oplus \cdots \oplus L_{r-1} \oplus L_r H^{-1}a \subseteq R^{r+1}$ whose matrix has the same form as that of $[\phi]$ in Case 1. By Lemma 1, there is a chain of isomorphisms $\psi : H^{-1}a \oplus H \rightarrow H^{-1}a \oplus R \rightarrow R \oplus R$ carrying $L_i H^{-1}a$ onto a submodule $N^r \subseteq R \oplus R$. By [1, Cor. 18.24], $(H^{-1}a \oplus H)/L_i H^{-1}a \cong R/L_i \oplus H$. Let $\Phi = (I_{r-1} \oplus \psi) \circ \phi_r : R^n \rightarrow R^m$. The matrix of $\Phi$ is $m \times n$ and the first $r = n$ rows are the same as $[\phi_r]$. The last row does not contribute any entries above the main diagonal. So, for each $j > i + 1$, the $i, j$th entry of $[\Phi]$ is 0. Since the cokernel of $[\Phi]$ is $S_A$ and $[\Phi]$ has the same number of rows and columns as $A, [\Phi]$ and $A$ are equivalent.

Case 4. Let $S_A = \oplus_{i=1}^r R/L_i \oplus H$, where $L_1, \ldots, L_r$ are integral ideals with $L_1 \subseteq \cdots \subseteq L_r$ and by replacing $H$ (if necessary) by an isomorphic copy, $H$ is an integral ideal. By [1, Thm. 18.20], there is an integral ideal $H_0$ with $H_0 H$ principal and $H_0 + H = R$. There is an $a \in R$ such that $J = (\prod_{i=1}^r L_i \cdot H_0)^{-1}a \subseteq H$. As in Case 1, there is an isomorphism $\phi_{r+1} : R^{r+1} \rightarrow L_1 \oplus \cdots \oplus L_{r-1} \oplus L_r H_0 \oplus J$. View $L_i \subseteq R$ for $1 \leq i \leq r, L_i H_0 \subseteq H_0$. As in Case 3, there is an isomorphism $\psi : H_0 \oplus H \rightarrow R \oplus R$ with $\psi(L_i H_0) = N \subseteq R \subseteq R$ and $R \oplus R / N \cong R \oplus R \oplus H$. Let $\Phi = (I_{r-1} \oplus \psi) \circ \phi_{r+1}$. Then $\Phi : R^{r+1} \rightarrow R^{r+1}$ and all the rows, except possibly the last two of $[\Phi]$, are the same as that of $[\phi]$ in Case 1. So, for each $j > i + 1$, the $i, j$th entry of $[\Phi]$ is 0. Since the cokernel of $\Phi$ is $S_A, [\Phi]$ and $A$ are equivalent.

Remark. While we could have given explicit formula for the entries in the matrices constructed in Cases 2, 3, and 4 as in Case 1, these entries are not canonically determined by $A$ as a result of the many choices made in their construction. In particular, the choices of $\alpha$ and $x_1, x_2$ in Lemma 1 are not canonically determined by the ideals $I, J$.

References


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