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p-TOPOLOGICAL CAUCHY COMPLETIONS

J. WIG and D. C. KENT

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ABSTRACT. The duality between "regular" and "topological" as convergence space properties extends in a natural way to the more general properties "p-regular" and "p-topological." Since earlier papers have investigated regular, p-regular, and topological Cauchy completions, we hereby initiate a study of p-topological Cauchy completions. A p-topological Cauchy space has a p-topological completion if and only if it is "cushioned," meaning that each equivalence class of nonconvergent Cauchy filters contains a smallest filter. For a Cauchy space allowing a p-topological completion, it is shown that a certain class of Reed completions preserve the p-topological property, including the Wyler and Kowalsky completions, which are, respectively, the finest and the coarsest p-topological completions. However, not all p-topological completions are Reed completions. Several extension theorems for p-topological completions are obtained. The most interesting of these states that any Cauchy-continuous map between Cauchy spaces allowing p-topological and p'-topological completions, respectively, can always be extended to a θ-continuous map between any p-topological completion of the first space and any p'-topological completion of the second.

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Introduction. The duality between "regular" and "topological" as convergence space properties was recognized only recently through the further study of certain "diagonal axioms" initially introduced by Kowalsky [7], Cook and Fischer [1]. The axioms $\mathbb{R}$ and $\mathbb{F}$ which characterize these two properties and reveal their duality are discussed in [4] and [6]. A subsequent characterization of "topological" using the notion of "neighborhood filter of a filter" (see [10]) makes this duality apparent using simpler and more familiar characterizations of these properties than those provided by the diagonal axioms.

The same dual relationship extends to generalizations of these properties which are called "p-regular" and "p-topological." The notion of a p-regularity was introduced in 1990 by Kent and Richardson [2] for reasons unrelated to the aforementioned duality. A convergence space (X,q) is *p-regular* relative to another convergence structure p on X if the p-closure of a filter q-converges to a point whenever the filter itself q-converges to that point. It was shown in [2] that p-regularity is significant; both because of its nice structural properties and also the fact that various convergence properties of interest can be described in terms of p-regularity for appropriate choices of p. The dual property, "p-topological ," was introduced and studied in a recent paper by Wilde and Kent [10].

At this point, it is not clear whether, or to what extent, the duality between regular and topological, or p-regular and p-topological, extends to the realm of Cauchy spaces and Cauchy completions. Regular Cauchy completions have been studied extensively (see [8], for example) and p-regular Cauchy completions were investigated in [3]. Topological completions (called "diagonal completions" because the Cauchy spaces allowing such completions were characterized by a diagonal condition) have been studied in [5]. What is so far missing is a study of p-topological Cauchy completions, which is of course the goal of this paper.

In Section 1, we review some properties of p-topological convergence spaces. Section 2 gives a simple characterization of p-topological Cauchy spaces which allow p-topological completions and shows that, for such spaces, the Wyler completion is the finest p-topological completion. Section 3 gives a diagonal characterization for Cauchy spaces which allow p-topological completions. The Reed family of completions [9] is revisited in Section 4, where it is shown that the Kowalsky completion is the coarsest p-topological completion. For a Cauchy space which allows a p-topological completion, all Reed completions "between" the Kowalsky and Wyler completions also preserve the p-topological property. However, not all p-topological completions are Reed completions and even when p is a topology and $(X,\mathscr{C})$ allows a topological completion, a p-topological completion of $(X,\mathscr{C})$ need not be topological. Section 5 gives some extension results for p-topological completions, which, for the most part, generalize extension theorems proved for topological completions in [5].

1. p-topological convergence spaces. For standard terminology and notation on convergence spaces, the reader is referred to [4] or [6]. Let $\mathbb{F}(X)$ denote the set of all (proper) filters on a set X; the ultrafilter generated by $x \in X$ is denoted by $\dot{x}$. If (X,q) is a convergence space, "$\mathscr{F}$ q-converges to x" is usually written as "$\mathscr{F} \xrightarrow{q} x$"; "$\mathrm{cl}_q$" and "$I_q$" denote the closure and the interior operators, respectively.

If X and J are arbitrary sets, a function $\lambda : J \to \mathbb{F}(X)$ is called a *filter selection function on* J. For $\mathscr{F} \in \mathbb{F}(J)$, we define $k\lambda\mathscr{F}$ to be the filter $\cup_{F\in\mathscr{F}} \cap_{x\in F} \lambda(x)$ in $\mathbb{F}(X)$; $k\lambda\mathscr{F}$ is called the *compression of* $\mathscr{F}$ *relative to* λ.

Let p and q denote two arbitrary convergence structures on a set X. We consider the following axioms:

$\mathbb{R}_{p,q}$**:** Let J be any set, $\psi : J \to X$, and let $\sigma : J \to \mathbb{F}(X)$ have the property that $\sigma(y) \xrightarrow{p} \psi(y)$ for all $y \in J$. If $\mathscr{F} \in \mathbb{F}(J)$ is such that $k\sigma\mathscr{F} \xrightarrow{q} x$, then $\psi(\mathscr{F}) \xrightarrow{q} x$.

$\mathbb{F}_{p,q}$**:** Let J be any set, $\psi : J \to X$, and let $\sigma : J \to \mathbb{F}(X)$ have the property that $\sigma(y) \xrightarrow{p} \psi(y)$ for all $y \in J$. If $\mathscr{F} \in \mathbb{F}(J)$ is such that $\psi(\mathscr{F}) \xrightarrow{q} x$, then $k\sigma\mathscr{F} \xrightarrow{q} x$.

Theorem 1.1 [4]**.** *Let (X,q) be a convergence space.*

(a) *(X,q) is regular if and only if (X,q) satisfies $\mathbb{R}_{q,q}$.*

(b) *(X,q) is topological if and only if (X,q) satisfies $\mathbb{F}_{q,q}$.*

Theorem 1.2 [10]**.** *Let (X,q) be a convergence space, p a convergence structure on X. Then (X,q) and p satisfies $\mathbb{R}_{p,q}$ if and only if (X,q) is p-regular. (i.e., $\mathscr{F} \xrightarrow{q} x \Rightarrow \mathrm{cl}_p \mathscr{F} \xrightarrow{q} x$).*

Definition 1.3 [10]**.** If (X,q) is a convergence space and p a convergence structure

on X, then (X,q) is defined to be *p-topological* if (X,q) and p satisfy $\mathbb{F}_{p,q}$.

A more intuitive characterization of p-topological convergence spaces is obtained via the following definition. Let (X,q) be a convergence space and $\mathcal{F} \in \mathbb{F}(X)$. Then $\mathcal{V}_q(\mathcal{F}) = \{A : I_q A \in \mathcal{F}\}$ is called the *q-neighborhood filter of* $\mathcal{F}$. If $\mathcal{F} = \dot{x}$, $\mathcal{V}_q(\dot{x}) = \mathcal{V}_q(X)$ is the *q-neighborhood filter* at x, obtained by intersecting all the filters which q-converges to x.

Theorem 1.4 [10]. *Let (X,q) be a convergence space and p a convergence structure on X. The following statements are equivalent:*

(a) *(X,q) is p-topological.*

(b) *$\mathcal{F} \xrightarrow{q} x$ implies that there is $\mathcal{G} \xrightarrow{q} x$ such that $\mathcal{F} \geq I_p\mathcal{G}$.*

(c) *$\mathcal{F} \xrightarrow{q} x$ implies that $\mathcal{V}_p\mathcal{F} \xrightarrow{q} x$.*

A comparison of Theorem 1.4(c) with the definition of p-regularity ($\mathcal{F} \xrightarrow{q} x$ implies that $\mathrm{cl}_p\,\mathcal{F} \xrightarrow{q} x$) further reveals the duality between "p-regular" and "p-topological". The following corollary to Theorem 1.4 gives an elegant characterization of "topology" in terms of convergence criteria.

Corollary 1.5. *A convergence space (X,q) is topological if and only if $\mathcal{F} \xrightarrow{q} x$ implies $\mathcal{V}_p\mathcal{F} \xrightarrow{q} x$.*

Proposition 1.6. *Let p and q be convergence structures on X.*

(a) *If (X,q) is p-topological, then $q \leq p$.*

(b) *If q is a topology, then (X,q) is p-topological if and only if $q \leq p$.*

Proof. (a) If $\mathcal{F} \xrightarrow{p} x$, then $\mathcal{F} \geq \mathcal{V}_p(x)$ and since $\dot{x} \xrightarrow{q} x$, $\mathcal{V}_p(\dot{x}) = \mathcal{V}_p(x) \xrightarrow{q} x$, which implies $\mathcal{F} \xrightarrow{q} x$. □

Structural properties of p-topological convergence spaces are studied in some detail in [10]. A few such results are summarized in the next proposition to convey additional insight into the nature of p-topological spaces.

Proposition 1.7. (a) *A subspace of a p-topological space is p'-topological, where p' denotes the restriction of p to the subspace.*

(b) *Let $(X,q) = \Pi_{i\in I}(X_i,q_i)$ and $(X,p) = \Pi_{i\in I}(X_i,p_i)$ be product convergence spaces and assume that each (X_i,q_i) is p_i-topological. Then (X,q) is p-topological.*

(c) *Let $\mathbb{C}(X)$ be the lattice of all convergence structures on X. Let $\mathcal{S} = \{q_i : i \in I\} \subseteq \mathbb{C}(X)$ and let $p \in \mathbb{C}(X)$ be such that (X,q_i) is p-topological, for all $i \in I$. If $q = \inf \mathcal{S}$ and $s = \sup \mathcal{S}$, then (X,q) and (X,s) are both p-topological.*

2. The fine p-topological completion. A Cauchy space $(X,\mathcal{C})$ is a set X with a collection $\mathcal{C}$ of filters on X satisfying

(1) for each $x \in X$, $\dot{x} \in \mathcal{C}$;

(2) if $\mathcal{G} \in \mathcal{C}$, $\mathcal{G} \leq \mathcal{F}$, then $\mathcal{F} \in \mathcal{C}$;

(3) if $\mathcal{F},\mathcal{G} \in \mathcal{C}$ and $\mathcal{F} \vee \mathcal{G}$ exists, then $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$.

Associated with a Cauchy space $(X,\mathcal{C})$ is a convergence structure $q_{\mathcal{C}}$ on X defined as follows: $\mathcal{F} \xrightarrow{q_{\mathcal{C}}} x$ if $\mathcal{F} \cap \dot{x} \in \mathcal{C}$. $(X,\mathcal{C})$ is said to be *Hausdorff* if $q_{\mathcal{C}}$-convergent filters have unique limits or, equivalently, if $\dot{x} \cap \dot{y} \in \mathcal{C}$ implies $x = y$. Two Cauchy filters

$\mathcal{F},\mathcal{G} \in \mathscr{C}$ are *equivalent* if $\mathcal{F} \cap \mathcal{G} \in \mathscr{C}$. For a Hausdorff Cauchy space $(X,\mathscr{C})$, the set of all Cauchy equivalence classes is denoted by X^*; the natural injection $\jmath : X \to X^*$ is given by $\jmath(x) = [\dot{x}]$, for all $x \in X$. If $(X,\mathscr{C})$ and $(Y,\mathscr{D})$ are Cauchy spaces, $f : (X,\mathscr{C}) \to (Y,\mathscr{D})$ is *Cauchy-continuous* if $f(\mathcal{F}) \in \mathscr{D}$ whenever $\mathcal{F} \in \mathscr{C}$. $(X,\mathscr{C})$ is *complete* if every $\mathcal{F} \in \mathscr{C}$ is $q_{\mathscr{C}}$-convergent.

A completion $((Y,\mathscr{D}),g)$ of a Cauchy space $(X,\mathscr{C})$ consists of a complete Cauchy space $(Y,\mathscr{D})$ along with a dense Cauchy embedding $g : (X,\mathscr{C}) \to (Y,\mathscr{D})$. The completion is *strict* if $\mathcal{A} \in \mathscr{D}$ implies that there is $\mathcal{F} \in \mathscr{C}$ such that $\mathrm{cl}_{q_{\mathscr{D}}}\, \jmath(\mathcal{F}) \leq \mathcal{A}$, and *Hausdorff* if $(Y,\mathscr{D})$ is Hausdorff. *Henceforth, we assume that all Cauchy spaces and Cauchy completions are Hausdorff unless otherwise indicated.*

A completion $((Y,\mathscr{D}),g)$ of a Cauchy space $(X,\mathscr{C})$ is in *standard form* if $Y = X^*$, $g = \jmath$, and $\jmath(\mathcal{F}) \xrightarrow{q_{\mathscr{D}}} [\mathcal{F}]$, for all $\mathcal{F} \in \mathscr{C}$. Reed [9] proved that every completion is equivalent to one in standard form and ,therefore, unless otherwise stated, we limit our attention to completions in standard form. It is well known that the finest completion of a Cauchy space $(X,\mathscr{C})$ is the *Wyler completion* $((X^*,\mathscr{C}^*),\jmath)$, generated by $\mathscr{C}^* = \{\jmath(\mathcal{F}) \cap [\dot{\mathcal{F}}] : \mathcal{F} \in \mathscr{C}\}$. The Wyler completion determines a completion functor on the category of Cauchy spaces and Cauchy-continuous maps.

Definition 2.1. Let $(X,\mathscr{C})$ be a Cauchy space, p a convergence structure on X. Then $(X,\mathscr{C})$ is p-topological (respectively, p-regular) if $\mathcal{F} \in \mathscr{C}$ implies $\mathcal{V}_p\mathcal{F} \in \mathscr{C}$ (respectively, $\mathrm{cl}_p\, \mathcal{F} \in \mathscr{C}$).

If $(X,\mathscr{C})$ is p-topological (respectively, p-regular), it is obvious that the associated convergence structure $q_{\mathscr{C}}$ is likewise p-topological (respectively, p-regular). It also follows by Proposition 1.6 that if $(X,\mathscr{C})$ is p-topological, then $q_{\mathscr{C}} \leq p$. The next proposition describes the *upper* and *lower p-topological modifications* of a Cauchy space. Analogous results for p-regular modifications are found in [3, Prop. 2.1].

Proposition 2.2. *Let $(X,\mathscr{C})$ be a Cauchy space and p a convergence structure on X. Then*

(a) *There is a finest p-topological Cauchy structure $\tau_p\mathscr{C}$ on X coaser than $\mathscr{C}$, generated by $\{\mathcal{V}_p^n\mathcal{F} : \mathcal{F} \in \mathscr{C}, n \in N\}$. (Note that $\tau_p\mathscr{C}$ does not generally preserve the Hausdorff property.)*

(b) *There is a coarsest p-topological Cauchy structure $\tau^p\mathscr{C}$ on X finer than $\mathscr{C}$ if and only if $\mathscr{C} \leq \tau_p\mathscr{C}_\delta$, where $\mathscr{C}_\delta$ denotes the discrete Cauchy structure on X. If it exists, $\tau^p\mathscr{C} = \{\mathcal{F} \in \mathbb{F}(X) : \mathcal{V}_p^n\mathcal{F} \in \mathscr{C}$, for all $n \in N\}$.*

(c) *If $\mathscr{C}$ is complete, so are $\tau_p\mathscr{C}$ and $\tau^p\mathscr{C}$ (assuming the latter exists).*

To define a p-topological or a p-regular completion of a Cauchy space $(X,\mathscr{C})$, it is necessary to first extend p to the completion set X^* in an appropriate way. We follow the procedure used to define p-regular completions in [3].

Definition 2.3. Let $(X,\mathscr{C})$ be a p-topological Cauchy space. Let p^* be the finest convergence structure on X^* subject to the following conditions:

(i) $\jmath(\mathcal{F})p^*$-converges to $\jmath(x)$ in X^* if and only if $\mathcal{F}$ p-converges to x in X;

(ii) If $\mathcal{F} \in \mathscr{C}$ is non-$q_{\mathscr{C}}$-convergent, then $\jmath(\mathcal{F}) \xrightarrow{p^*} [\mathcal{F}]$ in X^*.

$((X^*,\mathscr{D}),\jmath)$ is called a *p-topological completion* of $(X,\mathscr{C})$ if it is a completion such that

$(X^*,\mathcal{D})$ is p^*-topological.

Since $q_{\mathcal{C}} \le p$, it is clear that p^* is a Hausdorff convergence structure on X^*. The definition of p^* given here differs slightly from that in [3], but the two versions of p^* have the same ultrafilter convergence and, consequently, are equivalent for defining either p-regular or p-topological completions.

When referring to a "p-topological Cauchy space" or "p-topological completion", we always assume that p is an arbitrary convergence structure on X (subject, of course, to $q_{\mathcal{C}} \le p$), unless otherwise specified. Since the p-topological property is hereditary for Cauchy spaces as well as for convergence spaces, it is clear that any Cauchy space which allows a p-topological completion is itself p-topological.

If $(X,\mathcal{C})$ is p-topological, it follows directly from Definition 2.3 that the p^*-neighborhood filters for points in X^* are as follows: if $z = [\dot{x}] \in \jmath(X)$, then $\mathcal{V}_{p^*}(z) = \jmath(\mathcal{V}_p(x))$, and if $z \in X^*\backslash\jmath(X)$, then $\mathcal{V}_{p^*}(z) = \mathcal{M}_z \cap \dot{z}$, where $\mathcal{M}_z = \cap z$. From these observations, the next lemma follows easily.

Lemma 2.4. *If $(X,\mathcal{C})$ is p-topological and $M \subseteq X^*$, then $I_{p^*}M = \jmath(I_p\jmath^{-1}(M)) \cup \alpha(M)$, where $\alpha(M) = \{z \in M\backslash\jmath(X) : \jmath^{-1}(M) \in \mathcal{M}_z\}$. Thus, if $U \subseteq X$ is p-open, $\jmath(u) \subseteq X^*$ is p^*-open.*

A Cauchy space $(X,\mathcal{C})$ is said to be *cushioned* if, for each non-qc-convergent filter $\mathcal{F} \in \mathcal{C}$, $\mathcal{M}_{\mathcal{F}} = \cap[\mathcal{F}] \in \mathcal{C}$. Note that $\mathcal{M}_{\mathcal{F}} = \mathcal{M}_z$, where $z = [\mathcal{F}] \in X^*\backslash\jmath(X)$. In other words, a Cauchy space is cushioned if every equivalence class of nonconvergent Cauchy filters contains a smallest filter (or "cushion").

Proposition 2.5. *If $(X,\mathcal{C})$ is cushioned and p-topological, where p is a topology on X, then p^* is a topology on X^*.*

Proof. Let $z \in X^*$. By Definition 2.3, it follows that $\mathcal{V}_{p^*}(z) = \jmath(\mathcal{V}_p(x))$ if $z = \jmath(x)$, where $x \in X$, and $\mathcal{V}_{p^*}(z) = \mathcal{M}_z \cap \dot{z}$ if $z \in X^*\backslash\jmath(X)$. Since p is a topology, $\jmath(\mathcal{V}_p(x))$ has a base of p^*-open sets for each $x \in X$ by Lemma 2.4. If $z \in X^*\backslash\jmath(X)$, $\mathcal{M}_z \in z$ since $(X,\mathcal{C})$ is cushioned, and since $(X,\mathcal{C})$ is p-topological, $\mathcal{V}_p(\mathcal{M}_z) = \mathcal{M}_z$, which implies that $\mathcal{M}_z$ has a filter base of p-open sets. From this, we conclude that $\mathcal{V}_{p^*}(z) = \mathcal{M}_z \cap \dot{z}$ has a filter base of p^*-open sets. □

Lemma 2.6. *If $(X,\mathcal{C})$ allows a p-topological completion, then $(X,\mathcal{C})$ is cushioned.*

Proof. If $((X^*,\mathcal{D}),\jmath)$ is a p-topological completion of $(X,\mathcal{C})$, then, for each $z \in X^*\backslash\jmath(X)$, $\mathcal{V}_p(z) = \mathcal{M}_z \cap \dot{z}$ $q_{\mathcal{D}}$-converges to z and hence, $\jmath^{-1}(\mathcal{V}_{p^*}(z)) = \mathcal{M}_z \in z$. □

Theorem 2.7. *The Wyler completion $((X^*,\mathcal{C}^*),\jmath)$ of a cushioned, p-topological Cauchy space $(X,\mathcal{C})$ is p^*-topological.*

Proof. Let $A \in \mathcal{C}^*$. If $A \cap \dot{\jmath}(x) \in \mathcal{C}^*$ for some $x \in X$, then, by definition of $\mathcal{C}^*$, $\jmath(X) \in \mathcal{A}$. Furthermore, $M \in \mathcal{V}_{p^*}(\mathcal{A})$ if and only if $I_{p^*}M \in \mathcal{A}$, which is equivalent, by Lemma 2.4, to $I_p\jmath^{-1}(M) \in \jmath^{-1}(\mathcal{A})$. Thus, $\mathcal{V}_{p^*}\mathcal{A} = \jmath(\mathcal{V}_p\jmath^{-1}(\mathcal{A})) \in \mathcal{C}^*$ since $\mathcal{V}_p\jmath^{-1}(\mathcal{A}) \in \mathcal{C}$ by assumption. If $\mathcal{A} \cap \dot{z} \in \mathcal{C}^*$, where $z \in X^*\backslash\jmath(X)$, then $\mathcal{A} \ge \mathcal{M}_z \cap \dot{z}$. Since $(X,\mathcal{C})$ is cushioned and p-topological, $\mathcal{V}_p(\mathcal{M}_z) = \mathcal{M}_z$, from which it follows that $\mathcal{M}_z$ has a filter base of p-open sets (where $A \subseteq X$ is p-open if and only if $I_pA = A$). It follows easily that $\mathcal{V}_{p^*}(\mathcal{A}) \ge \mathcal{V}_{p^*}(z) = \mathcal{M}_z \cap \dot{z}$, where the latter filter has a base of p^*-open sets.

Thus, $\mathcal{V}_{p^*}(\mathcal{A}) \in \mathcal{C}^*$ and thus, $(X^*, \mathcal{C}^*)$ is p^*-topological. □

Corollary 2.8. *A p-topological space $(X, \mathcal{C})$ has a p-topological completion if and only if $(X, \mathcal{C})$ is cushioned.*

It is obvious from its construction that the Wyler completion is the largest completion of a given Cauchy space in standard form and hence, the finest Cauchy completion up to equivalence. It is, thus, clear from Theorem 2.7 that it is the finest p-topological completion of any cushioned, p-topological Cauchy space and thus, we refer to it as the *fine p-topological completion*. Additional p-topological completions are studied in Section 4.

3. The diagonal axioms $\mathbb{D}_p$. *topological completion* of a Cauchy space $(X, \mathcal{C})$ is one for which the convergence structure of the completion space is a topology. In [5], a diagonal condition $\mathbb{D}$ is defined for a Cauchy space $(X, \mathcal{C})$ which is necessary and sufficient for $(X, \mathcal{C})$ to have a topological completion. One might conjecture that a topological completion of $(X, \mathcal{C})$ is simply a p-topological completion for which $p = q_{\mathcal{C}}$, but this is in fact false, as is shown by an example in the next section. In this section, we define a more general diagonal axiom $\mathbb{D}_p$ which is necessary and sufficient for a Cauchy space to allow a p-topological completion.

Let $(X, \mathcal{C})$ be a Cauchy space and p a convergence structure on X such that $q_{\mathcal{C}} \le p$. Consider the following axioms.

$\mathbb{D}_p$: Let J be any set, $\psi : J \to X^*$, and let $\sigma : J \to \mathcal{C}$ be such that $\sigma(y) \xrightarrow{p} \jmath^{-1}\psi(y)$ if $\psi(y) \in \jmath(X)$, and $\sigma(y) \in z$ if $\psi(y) = z \in X^* \backslash \jmath(X)$. If $\mathcal{F} \in \mathbb{F}(J)$ is such that $\psi(\mathcal{F}) \ge \jmath(\mathcal{H}) \cap [\dot{\mathcal{H}}]$, for some $\mathcal{H} \in \mathcal{C}$, then $k\sigma\mathcal{F} \in \mathcal{C}$.

When $p = q_{\mathcal{C}}$, the axiom $\mathbb{D}_p$ reduces to the axiom $\mathbb{D}$ defined in [5].

Theorem 3.1. *If $(X, \mathcal{C})$ and p satisfy $\mathbb{D}_p$, then $(X, \mathcal{C})$ is cushioned and p-topological.*

Proof. First, we show that $(X, \mathcal{C})$ is cushioned. Let $z \in X^* \backslash \jmath(X)$ and let $z = \{\mathcal{F}_i : i \in A\}$ be an indexing of the members of z. Let $J = X \cup A$ and let $\psi : J \to X^*$ be defined by

$$\psi(x) = [\dot{x}], \quad x \in X \qquad \text{and} \qquad \psi(i) = z, \quad i \in A. \tag{3.1}$$

Let $\sigma : J \to \mathbb{F}(X)$ be defined by

$$\sigma(x) = \dot{x}, \quad x \in X \qquad \text{and} \qquad \sigma(x) = \mathcal{F}_i, \quad i \in A. \tag{3.2}$$

Let $\mathcal{F} \in z$ and let $\mathcal{G}$ be the filter on J with filter base $\{F \cup A : F \in \mathcal{F}\}$. Then $\psi(\mathcal{G}) = \jmath(\mathcal{F}) \cap \dot{z}$. So by $\mathbb{D}_p$, $k\sigma\mathcal{G} \in \mathcal{C}$. But clearly, $\mathcal{M}_z \ge k\sigma\mathcal{G}$, so $\mathcal{M}_z \in \mathcal{C}$.

Next, we show that $(X, \mathcal{C})$ is p-topological. Let $\mathcal{F} \in \mathcal{C}$ and let $J = \{(\mathcal{G}, x) : \mathcal{G}$ an ultrafilter, $\mathcal{G} \xrightarrow{p} x\}$. Let $\psi : J \to X^*$ be defined by $\psi(\mathcal{G}, x) = [\dot{x}]$, and $\sigma : J \to \mathcal{C}$ by $\sigma(\mathcal{G}, x) = \mathcal{G}$. Note that ψ is onto $\jmath(X)$. If $\mathcal{H} = \psi^{-1}\jmath(\mathcal{F})$ then $\mathcal{H} \in \mathbb{F}(J)$ and $\psi(\mathcal{H}) = \jmath(\mathcal{F}) \ge \jmath(\mathcal{F}) \cap [\dot{\mathcal{F}}]$. Thus, by $\mathbb{D}_p$, $k\sigma\mathcal{H} \in \mathcal{C}$. It remains to show that $\mathcal{F} \ge I_p(k\sigma\mathcal{H})$.

Let $F \in \mathcal{F}$; then $K = \psi^{-1}\jmath F$ is an arbitrary set in $\mathcal{H}$. For each pair $(\mathcal{G}, x) \in K$, choose $G_{(\mathcal{G},x)} \in \mathcal{G}$. Then $A = \bigcup\{G_{(\mathcal{G},x)} : (\mathcal{G}, x) \in K\}$ is an arbitrary set in $k\sigma\mathcal{H}$. It is easy to see that $A \in \mathcal{V}_p(x)$, for all $x \in F$. Thus, $F \subseteq I_p(A)$, which establishes that $\mathcal{F} \ge I_p(k\sigma\mathcal{H})$. □

Theorem 3.2. *$(X,\mathscr{C})$ allows a p-topological completion if and only if $(X,\mathscr{C})$ and p satisfy $\mathbb{D}_p$.*

Proof. It remains to show that a cushioned, p-topological Cauchy space $(X,\mathscr{C})$ satisfies $\mathbb{D}_p$. Let J, ψ, and σ be as stated in $\mathbb{D}_p$. Let $\mathscr{F} \in \mathbb{F}(J)$ be such that there is $\mathscr{H} \in \mathscr{C}$ with $\psi(\mathscr{F}) \geq \jmath(\mathscr{H}) \cap [\dot{\mathscr{H}}]$. Since $(X,\mathscr{C})$ is p-topological, there is $\mathscr{G} \in \mathscr{C}$ such that $\mathscr{H} \geq I_p(\mathscr{G})$. Given $G \in \mathscr{G}$, choose $F \in \mathscr{F}$ such that $\psi(F) \subseteq \jmath(I_pG) \cup \{[\mathscr{H}]\}$. Let $F = F_1 \cup F_2$, where $\psi(F_1) \subseteq \jmath(I_pG)$ and $\psi(F_2) = \{[\mathscr{H}]\}$. If $\mathcal{Y} \in F_1$, $\psi(\mathcal{Y}) \in \jmath(I_pG)$ and hence, $G \in \sigma(\mathcal{Y})$, which implies that $G \in \bigcap_{\mathcal{Y}\in F_1}\sigma(\mathcal{Y})$. If $\mathcal{Y} \in F_2$, $\psi(\mathcal{Y}) = [\mathscr{H}]$, which implies that $\sigma(\mathcal{Y}) \geq \mathscr{M}_{\mathscr{H}}$. It follows that $k\sigma\mathscr{F} \geq \mathscr{G} \cap \mathscr{M}_{\mathscr{H}} = \mathscr{M}_{\mathscr{H}}$, and $\mathscr{M}_{\mathscr{H}} \in \mathscr{C}$ since $(X,\mathscr{C})$ is cushioned. Thus, $\mathbb{D}_p$ is satisfied. □

Theorem 3.3. *If $(X,\mathscr{C})$ satisfies $\mathbb{D}$, the Wyler completion $((X^*,C^*),\jmath)$ is topological.*

Proof. Since $(X,\mathscr{C})$ satisfies $\mathbb{D}$, $q_{\mathscr{C}}$ is a topology by [5]. By Theorem 2.7, $(X^*,\mathscr{C}^*)$ is $(q_{\mathscr{C}})^*$-topological and it is evident that $(q_{\mathscr{C}})^* = q_{\mathscr{C}^*}$ is the convergence structure of the Wyler completion, which is topological by Proposition 2.5. □

At this point, we wish to correct an error on [5, p. 263], where it is falsely asserted that the Wyler completion of a Cauchy space satisfying $\mathbb{D}$ may fail to be topological, an obvious contradiction to Theorem 3.3. In [5], the *fine diagonal completion* of a Cauchy space $(X,\mathscr{C})$ is defined to be the topological modification of the Wyler completion. This is correct but redundant since, in this case, the topological modification of the Wyler completion is itself. Nonetheless, [5, Thm. 2.10] is correct as stated.

Theorem 3.4. (a) *If a completion $((X^*,\mathscr{D}),\jmath)$ of a Cauchy space $(X,\mathscr{C})$ is topological, then it is also $q_{\mathscr{C}}$-topological.*

(b) *A Cauchy space $(X,\mathscr{C})$ has a topological completion if and only if it has a $q_{\mathscr{C}}$-topological completion and $q_{\mathscr{C}}$ is a topology on X.*

Proof. (a) If the given completion is topological, then $q_{\mathscr{D}}$ is a topology on X^* and clearly, $q_{\mathscr{D}} \leq (q_{\mathscr{C}})^*$, where $(q_{\mathscr{C}})^*$ is also a topology by Proposition 2.5. By Proposition 1.6(b), $(X^*,\mathscr{D})$ is also $(q_{\mathscr{C}})^*$-topological and hence, the completion is $q_{\mathscr{C}}$-topological.

(b) If $(X,\mathscr{C})$ has a topological completion the desired implication follows by (a). Conversely, under the given assumptions $\mathbb{D}_{q_{\mathscr{C}}}$ is equivalent to $\mathbb{D}$ and so, a topological completion exists by Theorem 3.3. □

The converse of Theorem 3.4(a) would assert that, for a Cauchy space satisfying $\mathbb{D}$, every $q_{\mathscr{C}}$-topological completion is a topological completion. That this is false is shown in Section 4.

4. Reed's family of p-topological completions. Let $(X,\mathscr{C})$ be a Cauchy space and Λ the set of all selection functions $\lambda : X^* \to \mathbb{F}(X)$ which satisfy the following conditions: if $z = [\dot{x}] \in \jmath(X)$, $\lambda(z) = \dot{x}$; if $z \in X^*\backslash\jmath(X)$, there is $\mathscr{G} \in z$ such that $\lambda(z) \leq \mathscr{G}$. For every subset $\Gamma \subseteq \Lambda$, Reed [9] defined a completion $((X^*,\mathscr{C}_\Gamma),\jmath)$ in standard form and discussed various properties of these completions. We briefly review these definitions.

For an arbitrary $\lambda \in \Lambda$ and $F \subseteq X$, let $F^\lambda = \{z \in X^* : F \in \lambda(z)\}$, and if $\mathscr{F} \in \mathbb{F}(X)$, let $\mathscr{F}^\lambda$ be the filter on X^* generated by $\{F^\lambda : F \in \mathscr{F}\}$. Let $\mathscr{C}_\lambda = \{\mathscr{A} \in \mathbb{F}(X^*)$: there is $z \in X^*$ and

$\mathcal{F} \in z$ such that $\mathcal{A} \geq \mathcal{F}^{\lambda} \cap \dot{z}\}$. If $\Gamma \subseteq \Lambda$, let $\mathcal{C}_{\Gamma} = \bigcap\{\mathcal{C}_{\lambda} : \lambda \in \Gamma\}$ be the supremum (in the lattice of Cauchy structures on X^*) of the set $\{\mathcal{C}_{\lambda} : \lambda \in \Gamma\}$. For notational convenience, the associated convergence structures on X^* is designated by $q_{\lambda} = q_{\mathcal{C}_{\lambda}}$ and $q_{\Gamma} = q_{\mathcal{C}_{\Gamma}}$. If $\lambda, \mu \in \Lambda$, we define $\lambda \leq \mu$ to mean $\lambda(z) \leq \mu(z)$, for all $z \in X^*$.

Lemma 4.1. (a) *If $\lambda, \mu \in \Lambda$ and $\lambda \leq \mu$, then $\mathcal{C}_{\mu} \leq \mathcal{C}_{\lambda}$.*
(b) *If $\Gamma \subseteq \Delta \subseteq \Lambda$, then $\mathcal{C}_{\Gamma} \leq \mathcal{C}_{\Delta}$.*
(c) *If $\gamma \in \Gamma \subseteq \Lambda$ and $\lambda \in \Gamma$ implies $\gamma \leq \lambda$, then $\mathcal{C}_{\gamma} = \mathcal{C}_{\Gamma}$.*

Proof. (c) $\mathcal{C}_{\gamma} \leq \mathcal{C}_{\Gamma}$ follows by (b). Let $\mathcal{A} \in \mathcal{C}_{\gamma}$. Then there is $z \in X^*$ and $\mathcal{F} \in z$ such that $\mathcal{A} \geq \mathcal{F}^{\gamma} \cap [\dot{\mathcal{F}}]$. If $\lambda \in \Gamma$, then $\gamma \leq \lambda$ implies $\mathcal{F}^{\lambda} \leq \mathcal{F}^{\gamma}$ and hence, $\mathcal{A} \geq \mathcal{F}^{\lambda} \cap [\dot{\mathcal{F}}]$. Thus, $\mathcal{A} \in \mathcal{C}_{\lambda}$, for all $\lambda \in \Gamma$, and so, $\mathcal{A} \in \mathcal{C}_{\Gamma}$. □

Among the members of Reed's family of completions of a Cauchy space $(X,\mathcal{C})$ are two of particular interest. The *Wyler completion* $((X^*,\mathcal{C}^*),\jmath)$ is obtained by taking $\mathcal{C}^* = \mathcal{C}_{\omega}$, where ω is the smallest element in Λ, specified by $\omega(z) = \{X\}$, for all $z \in X^* \backslash \jmath(X)$. The *Kowalsky completion* $((X^*,C_{\sum}),\jmath)$, where $\sum = \{\gamma \in \Lambda : \gamma(z) \in z$, for all $z \in X^* \backslash \jmath(X)\}$. Another completion which is important for us is $((X^*,\mathcal{C}_{\sigma}),\jmath)$, where $\sigma(z) = \mathcal{M}_z$, for all $z \in X^* \backslash \jmath(X)$. If $(X,\mathcal{C})$ is cushioned (i.e., $\mathcal{M}_z \in z$, for all $z \in X^* \backslash \jmath(X)$), we observe that $\sigma \in \sum$ and $\lambda \in \sum$ implies $\sigma \leq \lambda$. Using Lemma 4.1 (c), we obtain the following.

Proposition 4.2. *If $(X,\mathcal{C})$ is a cushioned Cauchy space, then $((X^*,\mathcal{C}_{\sigma}),\jmath)$ is the Kowalsky completion.*

If $(X,\mathcal{C})$ has a p-topological completion belonging to Reed's completion family, we refer to this as a *Reed p-topological completion.* We proceed to characterize such completions. Let $\tilde{\Lambda} = \{\lambda \in \Lambda : \omega \leq \lambda \leq \sigma\}$.

Lemma 4.3. *Let $(X,\mathcal{C})$ be a p-topological Cauchy space and $\lambda \in \tilde{\Lambda}$. Then, for any $\mathcal{G} \in \mathbb{F}(X)$, $\mathcal{V}_{p^*}(\mathcal{G}^{\lambda}) \geq (\mathcal{V}_p(\mathcal{G}))^{\lambda}$.*

Proof. Let $M = B^{\lambda} \in (\mathcal{V}_p(\mathcal{G}))^{\lambda}$, where $B \in \mathcal{V}_p(\mathcal{G})$. Then $I_pB \in \mathcal{G}$ and so, $(I_pB)^{\lambda} \in \mathcal{G}^{\lambda}$. We show that $(I_pB)^{\lambda} \subseteq I_{p^*}(B^{\lambda})$, which yields the desired inequality. Let $z \in (I_pB)^{\lambda}$. If $z \in \jmath(X)$, then $z \in \jmath(I_pB) \subseteq \jmath(I_p\jmath^{-1}B^{\lambda}) \subseteq I_{p^*}B^{\lambda}$ by Lemma 2.4. If $z \in X^* \backslash \jmath(X)$, $I_pB \in \lambda(z)$ implies $B \in \lambda(z)$, and so, $B = \jmath^{-1}B^{\lambda} \in \mathcal{M}_z$, whence, $z \in \alpha(B^{\lambda}) \subseteq I_{p^*}B^{\lambda}$ by Lemma 2.4. □

Theorem 4.4. *Let $(X,\mathcal{C})$ be p-topological and cushioned and let $\Gamma \subseteq \tilde{\Lambda}$. Then $((X^*,\mathcal{C}_{\Gamma}),\jmath)$ is a p-topological completion.*

Proof. Let $\Gamma = \{\lambda_i : i \in I\}$. By assumption, $\lambda_i(z) \leq \mathcal{M}_z$, for all $i \in I$. Let $\mathcal{A} \in \mathcal{C}_{\Gamma}$. For each $i \in I$, there is $\mathcal{G}_i \in \mathcal{C}$ such that $\mathcal{A} \geq \mathcal{G}_i^{\lambda_i}$ and, by Lemma 4.3, $\mathcal{V}_{p^*}(\mathcal{A}) \geq \mathcal{V}_{p^*}(\mathcal{G}_i^{\lambda_i}) \geq (\mathcal{V}_p(\mathcal{G}_i))^{\lambda_i}$. $\mathcal{V}_p(\mathcal{G}_i) \in \mathcal{C}$ since $(X,\mathcal{C})$ is p-topological, and so, $\mathcal{V}_{p^*}(\mathcal{A}) \in \mathcal{C}_{\Gamma}$, thereby, establishing that $(X^*,\mathcal{C}_{\Gamma})$ is p^*-topological. □

Theorem 4.5. *If $((X^*,\mathcal{D}),\jmath)$ is a p-topological completion of $(X,\mathcal{C})$, then $\mathcal{C}_{\sigma} \leq \mathcal{D}$.*

Proof. Let $\mathcal{L} \in \mathcal{D}$. Then there is $\mathcal{M} \in \mathcal{D}$ such that $I_{p^*}(\mathcal{M}) \leq \mathcal{L}$ (see Theorem 1.4). Furthermore, $\mathcal{M} \in \mathcal{D}$ implies $\jmath^{-1}(\mathcal{M}) \in \mathcal{C}$. To complete the proof, we verify that $I_{p^*}\mathcal{M} \geq (\jmath^{-1}(\mathcal{M}))^{\sigma}$.

Let $M \in \mathcal{M}$ and let $z \in I_{p^*}M = \jmath I_p(\jmath^{-1}M) \cup \alpha_p M$, by Lemma 2.4. If $z = [\dot{x}]$, $x \in \jmath^{-1}M$ and $z \in (\jmath^{-1}M)^\sigma$. If $z \in \alpha_p M$, then $\jmath^{-1}M \in \mathcal{M}_z$ and hence, $z \in (\jmath^{-1}M)^\sigma$. Thus, $I_{p^*}M \subseteq (\jmath^{-1}M)^\sigma$, which establishes the desired filter inequality. Since $(\jmath^{-1}(\mathcal{M}))^\sigma \in \mathscr{C}_\sigma$, $\mathcal{L} \in \mathscr{C}_\sigma$ as desired. □

Corollary 4.6. *Let $(X,\mathscr{C})$ be p-topological and cushioned.*

(a) *The Kowalsky completion is the coarsest p-topological completion of $(X,\mathscr{C})$.*

(b) *The Reed p-topological completions (in standard form) are precisely those of the form $((X^*,\mathscr{C}_\Gamma),\jmath)$ where $\Gamma \subseteq \tilde{\Lambda}$.*

The Kowalsky completion of a p-topological, cushioned Cauchy space is also called the *coarse p-topological completion*. The significance of Reed p-topological completions is further demonstrated by the next theorem, which is a partial converse to Theorem 3.4(a).

Theorem 4.7. *Let $(X,\mathscr{C})$ be cushioned and $q_{\mathscr{C}}$-topological, where $q_{\mathscr{C}}$ is a topology. Then any Reed $q_{\mathscr{C}}$-topological completion is topological.*

Proof. First, let $\lambda \in \tilde{\Lambda}$. Note that, for $z \in X^*$, $\mathscr{V}_{q_\lambda}(z)$ is $(\mathscr{V}_{q_{\mathscr{C}}}(x))^\lambda$ if $z = [\dot{x}] \in \jmath(X)$, and $(\mathcal{M}_z)^\lambda \cap \dot{z}$ if $z \in X^* \backslash \jmath(X)$, where $\mathscr{V}_q c(x)$ and $\mathcal{M}_z$ have filter bases of $q_{\mathscr{C}}$-open sets. A simple argument shows that if $U \subseteq X$ is $q_{\mathscr{C}}$-open, then U^λ is q_λ-open. Thus, for every $z \in X^*$, $\mathscr{V}_{q_\lambda}(z)$ has a filter base of q_λ-open sets and therefore, q_λ is a topology and $((X^*,\mathscr{C}_\lambda),\jmath)$ is a topological completion.

Next, let $\Gamma \subseteq \tilde{\Lambda}$. Then q_Γ is the supremum in the lattice of convergence structures on X of $\{q_\lambda : \lambda \in \Lambda\}$. Since "topological" is an initial property for convergence structures, q_Γ is a topology and therefore, $((X^*,\mathscr{C}_\Gamma),\jmath)$ is a topological completion. □

Are all the p-topological completions of a cushioned, p-topological Cauchy space Reed completions? The answer is no, as the following example demonstrates. This example also shows that the full converse to Theorem 3.4(a) is false since there are Cauchy spaces $(X,\mathscr{C})$ which allow topological completions but for which there exist $q_{\mathscr{C}}$-topological completions that are not topological.

Example 4.8. Let X be the set Q of rational numbers and $\mathscr{C}$ the usual Cauchy structure for Q. We may identify X^* with the set R of real numbers, in which case $\jmath : Q \to R$ is the identity injection. Let $q^* = (q_{\mathscr{C}})^*$ denote the topology of the Wyler completion of $(X,\mathscr{C})$. Thus, for every $y \in R$, $\mathscr{V}_{q^*}(y)$ has a filter base of q^*-open sets of the form $((y-\epsilon, y+\epsilon) \cap Q) \cup \{y\}$ for all $\epsilon > 0$.

For each $n \in N$, let $\mathscr{G}_n$ be the filter on R generated by the collection of open intervals $\{(1/n-\epsilon, 1/n+\epsilon) : \epsilon > 0\}$. Let s be the pretopology on R with neighborhood filters defined as follows:

$$\mathscr{V}_s(y) = \mathscr{V}_{q^*}(y), \quad \text{if } y \in R \backslash \left\{\frac{1}{n} : n \in N\right\}, \tag{4.1}$$

$$\mathscr{V}_s\left(\frac{1}{n}\right) = \mathscr{G}_n, \quad \text{if } n \in N. \tag{4.2}$$

It is readily apparent that $\mathscr{V}_s(0)$ does not have a filter base of s-open sets. So, s is not a topology on R and hence, by Theorem 4.7, does not define a Reed-completion of $(X,\mathscr{C})$. However, for every $y \in R$, $\mathscr{V}_s(y)$ has a filter base of q^*-open sets, from which

it follows that $((R,s),\jmath)$ is a $q_{\mathscr{C}}$-topological completion of $(X,\mathscr{C})$.

Let $(X < \mathscr{C})$ be a Cauchy space which allows a p-topological completion. Since the Kowalsky completion of $(X,\mathscr{C})$ is strict, it follows, by Corollary 4.6(a), that every p-topological completion of $(X,\mathscr{C})$ is strict. In contrast, a Cauchy space which allows a p-regular completion has a unique strict p-regular completion. There are Cauchy spaces which allow a regular completion but no strict regular completion; for such a Cauchy space, the fine $q_{\mathscr{C}}$-regular completion is strict and hence, nonregular. Thus, p-regular and p-topological completions exhibit similar behaviour in this respect.

Proposition 4.9. *If $(X,\mathscr{C})$ allows both a p-regular completion $((X^*,\mathscr{D}),\jmath)$ and a p-topological completion $((X^*,\mathscr{D}),\jmath)$ then $\mathscr{D} \le \mathscr{D}'$.*

Proof. Since $((X^*,\mathscr{C}_\sigma),\jmath)$ is the coarsest p-topological completion (in standard form), it suffices to show that $\mathscr{C}_\sigma \subseteq \mathscr{D}$. If $\mathscr{A} \in \mathscr{C}_\sigma$, there is $\mathscr{G} \in \mathscr{C}$ such that $\mathscr{A} \ge \mathscr{G}^\sigma \cap [\dot{\mathscr{G}}] \ge \mathrm{cl}_{p^*}\,\jmath(\mathscr{G})$. The latter filter is in $\mathscr{D}$ since $\mathscr{D}$ is p-regular and therefore, $\mathscr{A} \in \mathscr{D}$. □

The existence of p-regular completions in Reed's completion family has yet to be investigated.

5. Extension theorems. Let $(X,\mathscr{C})$ and $(Y,\mathscr{D})$ be Cauchy spaces, and let $f:(X,\mathscr{C}) \to (Y,\mathscr{D})$ be Cauchy-continuous. Let $f^*: X^* \to Y^*$ be defined as follows:

$$f^*(z) = \begin{cases} [\dot{f(x)}], & \text{if } z = [\dot{x}],\ x \in X, \\ [\dot{y}], & \text{if } z \in X^* \setminus \jmath(X) \text{ and there is } \mathscr{F} \in z \text{ such that } f(\mathscr{F}) \cap \dot{y} \in \mathscr{D}, \\ [f(\mathscr{F})], & \text{if } z = [\mathscr{F}] \in X^* \setminus \jmath(X) \text{ and } f(\mathscr{F}) \text{ is non-}q_{\mathscr{D}}\text{-convergent.} \end{cases} \tag{5.1}$$

Then f^* is a well-defined function such that the canonical diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\jmath_X"'] & Y \arrow[d, "\jmath_Y"] \\ X^* \arrow[r, "f^*"] & Y^* \end{tikzcd}$$

commutes. The extension theorem for Wyler completion asserts that in the preceding diagram, f^* is Cauchy-continuous when X^* and Y^* are equipped with the Cauchy structures $\mathscr{C}^*$ and $\mathscr{D}^*$ of the respective Wyler completions.

Theorem 5.1. *Let $(X,\mathscr{C})$ be a cushioned, p-topological Cauchy space, let $(Y,\mathscr{D})$ be a cushioned, p'-topological Cauchy space, and let $f:(X,\mathscr{C}) \to (Y,\mathscr{D})$ be Cauchy-continuous. Then, for the fine p-topological completion $((X^*,\mathscr{C}^*),\jmath_X)$ of $(X,\mathscr{C})$ and the fine p'-topological completion $((Y^*,\mathscr{D}^*),\jmath_Y)$ of $(Y,\mathscr{D})$, all the functions in the following commutative diagram are Cauchy-continuous:*

$$\begin{tikzcd} (X,\mathscr{C}) \arrow[r, "f"] \arrow[d, "\jmath_X"'] & (Y,\mathscr{D}) \arrow[d, "\jmath_Y"] \\ (X^*,\mathscr{C}^*) \arrow[r, "f^*"] & (Y^*,\mathscr{D}^*). \end{tikzcd}$$

Theorem 5.1 is an obvious special case of the aforementioned Wyler Extension Theorem. It is worth noting that no assumptions are needed concerning the behavior of $f : (X,p) \to (Y,p')$. This same absence of a necessary relationship between p and p' is observed again in Theorem 5.8.

Reed [9] proved that if $f : (X,\mathscr{C}) \to (Y,\mathscr{D})$ is Cauchy-continuous, where $(Y,\mathscr{D})$ is complete and regular, then, for any $\Gamma \subseteq \Lambda$, there is a unique, Cauchy-continuous extension $f' : (X^*,\mathscr{C}_\Gamma) \to (Y,\mathscr{D})$. In particular, this result applies to the Kowalsky completion, and this, along with Corollary 4.6(a), gives us another p-topological extension theorem.

Theorem 5.2. *Let $(X,\mathscr{C})$ be cushioned and p-topological, and let $((X^*,\mathscr{C}'),\jmath)$ be any p-topological completion. Let $f : (X,\mathscr{C}) \to (Y,\mathscr{D})$ be Cauchy-continuous, where $(Y,\mathscr{D})$ is complete and regular. Then there is a unique, Cauchy-continuous map $f' : (X^*,\mathscr{C}') \to (Y,\mathscr{D})$ which makes the following diagram commute:*

$$\begin{tikzcd}
(X,\mathscr{C}) \arrow[dr, "f"] \arrow[dd, "\jmath"'] & \\
& (Y,\mathscr{D}) \\
(X^*,\mathscr{C}') \arrow[ur, "f'"'] &
\end{tikzcd}$$

In the study of topological completions in [5], it was shown that if $f : (X,\mathscr{C}) \to (Y,\mathscr{D})$ is Cauchy-continuous, where $(X,\mathscr{C})$ and $(Y,\mathscr{D})$ both satisfy the axiom $\mathbb{D}$, and $((X^*,\tilde{\mathscr{C}}),\jmath_X)$ and $((Y^*,\tilde{\mathscr{D}}),\jmath_Y)$ denote the coarse completions of $(X,\mathscr{C})$ and $(Y,\mathscr{D})$, respectively, then $f^* : (X^*,\tilde{\mathscr{C}}) \to (Y^*,\tilde{\mathscr{D}})$ is θ-continuous. We wish to generalize this result to p-topological completions and our first task is to define "θ-continuity" in the setting of convergence spaces (or complete Cauchy spaces, which amount to the same thing). Such a definition has already been given in [2] and can be reformulated as follows.

Definition 5.3. Let (X,q) and (Y,s) be convergence spaces. $f : (X,q) \to (Y,s)$ is *θ-continuous* if, for each $\mathscr{G} \xrightarrow{q} x$ and $n \in N$, there is $\mathscr{H} \xrightarrow{s} f(x)$ and $m \in N$ such that $f(\mathrm{cl}_q^n \mathscr{G}) \geq \mathrm{cl}_s^m \mathscr{H}$.

Let $(X,\mathscr{C})$ be a Cauchy space. Using the notation of [3], we define, for any $A \subseteq X$, $\theta_{\mathscr{C}} A = \{z \in X^* \setminus \jmath(X) : \mathscr{M}_z$ has a trace on $A\}$. If $((X^*,\mathscr{D}),\jmath)$ is any completion of $(X,\mathscr{C})$ in standard form, it is obvious that $\jmath(\mathrm{cl}_{q_{\mathscr{C}}} A) \cup \theta_{\mathscr{C}} A \subseteq \mathrm{cl}_{q_{\mathscr{D}}} \jmath(A)$.

Recall that $((X^*,\mathscr{C}_\sigma),\jmath)$ denotes the Kowalsky completion of a cushioned Cauchy space $(X,\mathscr{C})$, and q_σ the associated convergence structure on X^*. We omit the straightforward proofs of the next lemmas.

Lemma 5.4. *Let $(X,\mathscr{C})$ be a cushioned Cauchy space, $A \subseteq X$, and $n \in N$. Then $\mathrm{cl}_{q_\sigma}^n \jmath(A) = \jmath(\mathrm{cl}_{q_{\mathscr{C}}}^n A) \cup \theta_{\mathscr{C}}(\mathrm{cl}_{q_{\mathscr{C}}}^{n-1} A)$.*

Corollary 5.5. *Let $(X,\mathscr{C})$ be cushioned and p-topological, and let $((X^*,\mathscr{D}),\jmath)$ be a p-topological completion of $(X,\mathscr{C})$. If $A \subseteq X$ and $n \in N$, then $\mathrm{cl}_{q_{\mathscr{D}}}^n \jmath(A) = \jmath(\mathrm{cl}_q^n A) \cup$*

$\theta_{\mathscr{C}}(\mathrm{cl}_q^{n-1} A) = \mathrm{cl}_{q_\sigma}^n \jmath(A)$.

Let $f : (X,\mathscr{C}) \to (Y,\mathscr{D})$ be Cauchy-continuous, where $(X,\mathscr{C})$ is cushioned and p-topological, and $(Y,\mathscr{D})$ is cushioned and p'-topological. (No relationship is assumed between the convergence structures p on X and p' on Y.) As usual, $((X^*,\mathscr{C}_\sigma),\jmath_X)$ denotes the Kowalsky completion of $(X,\mathscr{C})$, and we use $((Y^*,\mathscr{D}_{\sigma'}),\jmath_Y)$ to denote the Kowalsky completion of $(Y,\mathscr{C})$, where $\sigma' : Y^* \to \mathbb{F}(Y)$ is the appropriate selection function which characterizes this completion in accordance with Proposition 4.2.

Lemma 5.6. *Let $f : (X,\mathscr{C}) \to (Y,\mathscr{D})$ be Cauchy-continuous. Under the assumptions and in the notation of the preceding paragraph, it follows that, for any $A \subseteq X$ and $n \in N$, $f^*(\mathrm{cl}_{q_\sigma}^n \jmath_X(A)) \subseteq \mathrm{cl}_{q'_\sigma}^n \jmath_Y f(A)$.*

The next result generalizes [5, Thm. 2.11].

Theorem 5.7. *Let $f : (X,\mathscr{C}) \to (Y,\mathscr{D})$ be Cauchy-continuous, where $(X,\mathscr{C})$ is cushioned and p-topological, and $(Y,\mathscr{D})$ is cushioned and p'-topological. Then $f^* : (X^*,\mathscr{C}_\sigma) \to (Y^*,\mathscr{D}_{\sigma'})$ is θ-continuous.*

Proof. Let $\mathscr{A} \xrightarrow{q_\sigma} z$ in X^* and $n \in N$. Then there is $\mathscr{G} \in \mathscr{C}$ such that $\mathscr{A} \geq \mathscr{G}^\sigma \cap \dot{z}$. If $z = [\dot{x}]$ where $x \in X$, then $\mathscr{G} \xrightarrow{q_{\mathscr{C}}} x$, which implies that $f(\mathscr{G}) \xrightarrow{q_{\mathscr{D}}} f(x)$ and $\jmath_Y f(\mathscr{G}) \xrightarrow{q_{\sigma'}} [\dot{f(x)}] = f^*(z)$. If $z \in X^* \backslash \jmath(X)$ and $f(\mathscr{G}) \xrightarrow{q_{\mathscr{D}}} y \in Y$, then $\jmath_Y f(\mathscr{G}) \xrightarrow{q'_\sigma} [\dot{y}] = f^*(z)$. If $f(\mathscr{G})$ is non-$q_{\mathscr{D}}$-convergent, $\jmath_Y f(\mathscr{G}) \xrightarrow{q'_\sigma} f^*(z)$. Thus, in every case, $\jmath_Y f(\mathscr{G}) \xrightarrow{q'_\sigma} f^*(z)$. Therefore, $\mathrm{cl}_{q_\sigma}^n \mathscr{A} \geq \mathrm{cl}_{q_\sigma}^n \mathscr{G}^\sigma \geq \mathrm{cl}_{q_\sigma}^{n+1} \jmath_X(\mathscr{G})$ and by Lemma 5.6, $f^*(\mathrm{cl}_{q_\sigma}^n \jmath_X \mathscr{G}) \geq \mathrm{cl}_{q'}^{n+1} \times \jmath_Y f(\mathscr{G})$. Since $\jmath_Y f(\mathscr{G})$ is q'_σ-convergent to $f^*(z)$, the θ-continuity of f^* is established. □

Theorem 5.8. *Under the assumptions of the preceding theorem, let $((X^*,\tilde{\mathscr{C}}),\jmath_X)$ be any p-topological completion of $(X,\mathscr{C})$ and let $((Y^*,\hat{\mathscr{C}}),\jmath_Y)$ be any p'-topological completion of $(Y,\mathscr{D})$. Then $f^* : (X^*,\tilde{\mathscr{C}}) \to (Y^*,\hat{\mathscr{C}})$ is θ-continuous.*

Proof. The proof is accomplished by making suitable modifications in the proof of Theorem 5.7. Let $\tilde{q}$ be the convergence structure on X^* induced by $\tilde{C}$ and $\hat{q}$ the convergence structure on Y^* induced by $\hat{\mathscr{C}}$. Let $\mathscr{A} \xrightarrow{\tilde{q}} z$ in X^* and $n \in N$. Since $q_\sigma \leq \tilde{q}$, $\mathscr{A} \xrightarrow{q_\sigma} z$ and so, there is $\mathscr{G} \in \mathscr{C}$ such that $\mathscr{A} \geq \mathscr{G}^\sigma \cap \dot{z}$; furthermore, as in the preceding proof, $\jmath_Y f(\mathscr{G}) \xrightarrow{q'_\sigma} f^*(z)$. From this, it follows that $\jmath_Y f(\mathscr{G}) \xrightarrow{\hat{q}} f^*(z)$, since a filter on Y^* containing $\jmath_Y Y$ exhibits the same convergence relative to any completion of $(Y,\mathscr{D})$ in standard form. Thus, $f^*(\mathrm{cl}_{\tilde{q}}^n \mathscr{A}) \geq f^*(cl_{q_\sigma}^n \mathscr{A}) \geq f^*(\mathrm{cl}_{q_\sigma}^{n+1} \jmath_X(\mathscr{G})) \geq \mathrm{cl}_{q'_\sigma}^{n+1} \jmath_Y(f(\mathscr{G})) = \mathrm{cl}_{\hat{q}}^{n+1} \jmath_Y f(\mathscr{G})$, where the last equality follows by Corollary 5.5. This completes the proof of the theorem. □

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Wig and Kent: Department of Pure and Applied Mathematics, Washington State University, Pullman, WA 99164-3113, USA