ON θ -GENERALIZED CLOSED SETS

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ABSTRACT. The aim of this paper is to study the class of θ -generalized closed sets, which is properly placed between the classes of generalized closed and θ -closed sets. Furthermore, generalized Λ -sets [16] are extended to θ -generalized Λ -sets and R_0 -, $T_{1/2}$ - and T_1 -spaces are characterized. The relations with other notions directly or indirectly connected with generalized closed sets are investigated. The notion of TGO-connectedness is introduced.

Keywords and phrases. θ -generalized closed, θ -closure, Λ -set, TGO-connected.

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1. Introduction. The first step of generalizing closed sets was done by Levine in 1970 [15]. He defined a set A to be generalized closed if its closure belongs to every open superset of A and introduced the notion of $T_{1/2}$ -spaces, which is properly placed between T_0 -spaces and T_1 -spaces. Dunham [10] proved that a topological space is $T_{1/2}$ if and only if every singleton is open or closed. In [13], Khalimsky, Kopperman, and Meyer proved that the digital line is a typical example of a $T_{1/2}$ -space.

Ever since, general topologists extended the study of generalized closed sets on the basis of generalized open sets: regular open, α -open [20], semi-open [14], semi-preopen [1], preopen [19], θ -open [26], δ -open [26], etc.

Extensive research on generalizing closedness was done in recent years as the notions of semi-generalized closed, generalized semi-closed, generalized α -closed, α -generalized closed, generalized semi-preclosed, regular generalized closed, γ -g-closed and (γ, γ') -g-closed sets were investigated [2, 3, 6, 7, 11, 18, 17, 22, 23, 24, 25].

Recently, in [8], Ganster and the first author of this paper defined δ -generalized closed sets and introduced the notion of $T_{3/4}$ -spaces, which is properly placed between T_1 -spaces and $T_{1/2}$ -spaces. They proved that the digital line is $T_{3/4}$.

The aim of this paper is to continue the study of generalized closed sets, this time via the θ -closure operator defined in [26] and characterize $T_{1/2}$ -spaces and T_1 -spaces in terms of θ -generalized closed sets. Via θ -closure operator, we extend the class of generalized Λ -sets to the class of θ -generalized Λ -sets and study some new characterizations of R_0 -spaces and T_1 -spaces.

2. Preliminaries concerning generalized closed sets. Throughout this paper, we consider spaces on which no separation axioms are assumed unless explicitly stated. The topology of a given space X is denoted by τ and (X, τ) is replaced by X if there is no chance for confusion. For $A \subseteq X$, the closure and the interior of A in X are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. Sometimes, when there is no chance for

confusion, $\bar{A}$ stands for $\text{Cl}(A)$. The θ -interior [26] of a subset A of X is the union of all open sets of X whose closures are contained in A , and is denoted by $\text{Int}_\theta(A)$. The subset A is called θ -open [26] if $A = \text{Int}_\theta(A)$. The complement of a θ -open set is called θ -closed. Alternatively, a set $A \subset (X, \tau)$ is called θ -closed [26] if $A = \text{Cl}_\theta(A)$, where $\text{Cl}_\theta(A) = \{x \in X : \bar{U} \cap A \neq \emptyset, U \in \tau \text{ and } x \in U\}$. The family of all θ -open sets forms a topology on X and is denoted by τ_θ . We use the name CO-set for sets whose closure is open.

OBSERVATION 2.1. (i) If A is preopen, then $\text{Cl}_\alpha(A) = \text{Cl}(A) = \text{Cl}_\theta(A)$.

(ii) Every CO-set is preopen.

(iii) Every dense subset is a CO-set.

(iv) Every subset of a space (X, τ) is a CO-set if and only if (X, τ) is locally indiscrete.

DEFINITION 1. A subset A of a space (X, τ) is called

- (1) a *generalized closed set* (= *g-closed*) [15] if $A \subseteq U$ and $U \in \tau$ implies that $\bar{A} \subseteq U$,
- (2) a *semi-generalized closed set* (= *sg-closed*) [4] if $A \subseteq U$ and U is semi-open implies that ${}_s\text{Cl}(A) \subseteq U$,
- (3) a *generalized α -closed set* (= *g α -closed*) [17] if $A \subseteq U$ and U is α -open implies that $\text{Cl}_\alpha(A) \subset U$,
- (4) a *generalized semi-closed set* (= *gs-closed*) [2] if $A \subseteq U$ and $U \in \tau$ implies that ${}_s\text{Cl}(A) \subseteq U$,
- (5) an *α -generalized closed set* (= *α g-closed*) [18] if $A \subseteq U$ and $U \in \tau$ implies that $\text{Cl}_\alpha(A) \subset U$,
- (6) a *generalized semi-preclosed set* (= *gsp-closed*) [7] if $A \subseteq U$ and $U \in \tau$ implies that ${}_{\text{sp}}\text{Cl}(A) \subseteq U$,
- (7) a *regular generalized closed set* (= *r-g-closed*) [23] if $A \subseteq U$ and U is regular open implies that $\bar{A} \subseteq U$.

DEFINITION 2. A topological space (X, τ) is called

- (1) R_0 -space [5] if the closures of every two different points are either disjoint or coincide,
- (2) R_1 -space [5] if every two different points, with distinct closures, have disjoint neighborhoods,
- (3) $T_{1/2}$ -space [15] if every g-closed set is closed, (= every singleton is open or closed [10]),
- (4) *kc-space* [27] if every compact set is closed.

DEFINITION 3. Recall that a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

- (1) *g-continuous* [3] if $f^{-1}(V)$ is g-closed in (X, τ) for every closed set V of (Y, σ) ,
- (2) *semi-continuous* [14] if $f^{-1}(V)$ is semi-open in (X, τ) for every open set V of (Y, σ) ,
- (3) *strongly θ -continuous* [21] if, for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(\bar{U}) \subseteq V$.

3. Basic properties of θ -generalized closed sets

DEFINITION 4. A subset A of a topological space (X, τ) is called *θ -generalized closed* (= *θ -g-closed*) if $\text{Cl}_\theta(A) \subseteq U$, whenever $A \subseteq U$ and U is open in (X, τ) .

We denote the family of all θ -generalized closed subsets of a space (X, τ) by $\text{TGC}(X, \tau)$.

The next two results together with the examples following them show that the class of θ -generalized closed sets is properly placed between the classes of g -closed and θ -closed sets.

OBSERVATION 3.1. *Every θ -closed set is θ -generalized closed.*

EXAMPLE 3.2. Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a, b\}, X\}$. Set $A = \{a, c\}$. Since the only open superset of A is X , A is clearly θ -generalized closed. But it is easy to see that A is not θ -closed. In fact, it is not even semi-closed since its complement $\{b\}$ has empty interior.

OBSERVATION 3.3. *Every θ -generalized closed set is g -closed and hence α g -closed, gs -closed, and r - g -closed.*

EXAMPLE 3.4. Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Set $A = \{c\}$. Clearly, A is closed and hence g -closed. Next, set $U = \{a, c\}$. Note that $X = \text{Cl}_\theta(A) \not\subseteq U \in \tau$. Thus, A is not θ -generalized closed.

The following diagram is an enlargement of a Diagram from [7].

OBSERVATION 3.5. *Let (X, τ) be a regular space (not necessarily even T_0). Then a subset A of X is θ -generalized closed if and only if A is generalized closed.*

LEMMA 3.6 [12, Thm. 3.1(d), Thm. 3.6(d)]. *For a space (X, τ) , the following conditions are equivalent*

- (1) X is an R_1 -space;
- (2) for each $x \in X$, $\text{Cl}\{x\} = \text{Cl}_\theta\{x\}$;
- (3) for each compact set $A \subseteq X$, $\text{Cl}(A) = \text{Cl}_\theta(A)$.

PROPOSITION 3.7. *If (X, τ) is R_1 , then a compact subset K of X is g -closed if and only if K is θ - g -closed.*

PROPOSITION 3.8. *Let A be a preopen subset of a topological space (X, τ) . Then the*

following conditions are equivalent

- (1) A is θ - g -closed;
- (2) A is g -closed;
- (3) A is αg -closed.

PROOF. Follows easily from Observation 2.1(i) (note that a preopen g -closed set is a CO-set). $\square$

LEMMA 3.9. *If A and B are subsets of a topological space (X, τ) , then $\text{Cl}_\theta(A \cup B) = \text{Cl}_\theta(A) \cup \text{Cl}_\theta(B)$ and $\text{Cl}_\theta(A \cap B) \subseteq \text{Cl}_\theta(A) \cap \text{Cl}_\theta(B)$.*

- PROPOSITION 3.10.** (i) *A finite union of θ - g -closed sets is always a θ - g -closed set.*
(ii) *A countable union of θ - g -closed sets need not be a θ - g -closed set.*
(iii) *A finite intersection of θ - g -closed sets may fail to be a θ - g -closed set.*

PROOF. (i) Let $A, B \in \text{TGC}(X)$. Let $U \in \tau$ such that $A \cup B \subseteq U$. By Lemma 3.9, $\text{Cl}_\theta(A \cup B) = \text{Cl}_\theta(A) \cup \text{Cl}_\theta(B) \subseteq U \cup U = U$ since A and B are θ - g -closed. Hence, $A \cup B$ is θ - g -closed.

(ii) Let X be the real line with the usual topology. Since X is regular, by Observation 3.5, every singleton in X is θ - g -closed. Set $A = \bigcup_{i=2}^{\infty} \{1/i\}$. Clearly, A is a countable union of θ -generalized closed sets but A is not θ -generalized closed since $A \subseteq (0, 1)$ and $0 \in \text{Cl}_\theta(A)$.

(iii) Let $X = \{a, b, c, d, e\}$ and let $\tau = \{\emptyset, \{a, b\}, \{c\}, \{a, b, c\}, X\}$. Set $A = \{a, c, d\}$ and $B = \{b, c, e\}$. Clearly, A and B are θ -generalized closed sets since X is their only open superset. But $C = \{c\} = A \cap B$ is not θ -generalized closed since $C \subseteq \{c\} \in \tau$ and $\text{Cl}_\theta(C) = \{c, d, e\} \not\subseteq \{c\}$. $\square$

PROPOSITION 3.11. *The intersection of a θ -generalized closed set and a θ -closed set is always θ -generalized closed.*

PROOF. Let A be θ -generalized closed and let F be θ -closed. Let U be an open set such that $A \cap F \subseteq U$. Set $G = X \setminus F$. Then $A \subseteq U \cup G$. Since G is θ -open, $U \cup G$ is open and since A is θ -generalized closed, $\text{Cl}_\theta(A) \subseteq U \cup G$. Now, by Lemma 3.9, $\text{Cl}_\theta(A \cap F) \subseteq \text{Cl}_\theta(A) \cap \text{Cl}_\theta(F) = \text{Cl}_\theta(A) \cap F \subseteq (U \cup G) \cap F = (U \cap F) \cup (G \cap F) = (U \cap F) \cup \emptyset \subseteq U$. $\square$

PROPOSITION 3.12. *Let $B \subseteq H \subseteq (X, \tau)$ and $(\text{Cl}_\theta)_H(B)$ denote the θ -closure of B in the subspace $(H, \tau|_H)$. Then*

- (i) $(\text{Cl}_\theta)_H(B) \subseteq \text{Cl}_\theta(B) \cap H$ holds.
- (ii) *If H is open in (X, τ) , then $(\text{Cl}_\theta)_H(B) \supset \text{Cl}_\theta(B) \cap H$ holds.*

THEOREM 3.13. *Let $B \subseteq H \subseteq (X, \tau)$.*

- (i) *If B is θ - g -closed relative to H (i.e., $B \in \text{TGC}(H, \tau|_H)$), $H \in \text{TGC}(X)$, and $H \in \tau$, then $B \in \text{TGC}(X)$.*
- (ii) *If B is θ - g -closed in (X, τ) , then B is θ - g -closed relative to H (i.e., $B \in \text{TGC}(H, \tau|_H)$).*

PROOF. (i) Let $B \subseteq U$, where $U \in \tau$. Then $B \subseteq H \cap U$ and, moreover, $(\text{Cl}_\theta)_H(B) \subseteq H \cap U$ due to assumption. By Proposition 3.12(ii), $H \cap \text{Cl}_\theta(B) \subseteq H \cap U \subseteq U$. Using the last inclusion, it follows that $H \subseteq H \cup (X \setminus \text{Cl}_\theta(B)) = (H \cap \text{Cl}_\theta(B)) \cup (X \setminus \text{Cl}_\theta(B)) \subseteq U \cup (X \setminus \text{Cl}_\theta(B))$. Since $\text{Cl}_\theta(B)$ is a closed set, $U \cup (X \setminus \text{Cl}_\theta(B))$ is open and thus since $H \in \text{TGC}(X)$, $\text{Cl}_\theta(H) \subseteq U \cup (X \setminus \text{Cl}_\theta(B))$. Now, $\text{Cl}_\theta(B) \subseteq \text{Cl}_\theta(H) \subseteq U \cup (X \setminus \text{Cl}_\theta(B))$. From the

last inclusion, it follows that $\text{Cl}_\theta(B) \subseteq U$ or, equivalently, $B \in \text{TGC}(X)$.

(ii) Let V be an open set of $(H, \tau \mid H)$ such that $B \subset V$. Then there exists an open set $G \in \tau$ such that $G \cap H = V$. Since $B \subseteq G \cap H \subseteq G$ and $B \in \text{TGC}(X)$, $\text{Cl}_\theta(B) \subseteq G$. By Proposition 3.12(i), $(\text{Cl}_\theta)_H(B) \subseteq \text{Cl}_\theta(B) \cap H \subseteq G \cap H \subseteq V$. Therefore, B is θ -g-closed relative to H . $\square$

EXAMPLE 3.14. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$. Then $\{\emptyset, X\}$ is the set of all θ -closed sets of (X, τ) and $\text{TGC}(X, \tau) = \{\emptyset, \{b, c\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}, \{a, b, c\}, X\}$. Let $H = \{b, c, d\}$ be a set of X . Then, $\tau \mid H = \{\emptyset, \{b\}, \{c, d\}, H\}$. Note that $\{\emptyset, \{b\}, \{c, d\}, H\}$ is the set of all θ -closed sets of $(H, \tau \mid H)$ and $\text{TGC}(H, \tau \mid H) = \mathcal{P}(H)$. The subset $\{b\}$ of H is θ -g-closed relative to H and H is not open (i.e., $\{b\} \in \text{TGC}(H, \tau \mid H)$, $H \notin \tau$) and $H \in \text{TGC}(X, \tau)$. However, $\{b\} \notin \text{TGC}(X, \tau)$.

EXAMPLE 3.15. Let (X, τ) be the space in the example above. Set $H = \{a, c, d\}$. Clearly, H is open in (X, τ) and H is not θ -generalized closed in (X, τ) . But $B = \{a, c\}$ is θ -generalized closed relative to H . However, B is not θ -generalized closed in (X, τ) .

4. Characterizations of $T_{1/2}$ -spaces, T_1 -spaces and R_0 -spaces

THEOREM 4.1. A space (X, τ) is a $T_{1/2}$ -space if and only if every θ -generalized closed set is closed.

PROOF.

NECESSITY. Let $A \subseteq X$ be θ -generalized closed. By Observation 3.3, A is g-closed. Since X is a $T_{1/2}$ -space, A is closed.

SUFFICIENCY. Let $x \in X$. If $\{x\}$ is not closed, then $B = X \setminus \{x\}$ is not open and thus the only superset of B is X . Trivially, B is θ -generalized closed. By (2), B is closed or, equivalently, $\{x\}$ is open. Thus, every singleton in X is open or closed. Hence, in the notion of [6, Thm. 6.2(i)], X is a $T_{1/2}$ -space. $\square$

LEMMA 4.2. Let $A \subseteq (X, \tau)$ be θ -generalized closed. Then $\text{Cl}_\theta(A) \setminus A$ does not contain a nonempty closed set.

THEOREM 4.3. A space (X, τ) is a T_1 -space if and only if every θ -generalized closed set is θ -closed.

PROOF.

NECESSITY. Let $A \subseteq X$ be θ -generalized closed and let $x \in \text{Cl}_\theta(A)$. Since X is T_1 , $\{x\}$ is closed and thus by Lemma 4.2, $x \notin \text{Cl}_\theta(A) \setminus A$. Since $x \in \text{Cl}_\theta(A)$, then $x \in A$. This shows that $\text{Cl}_\theta(A) \subseteq A$ or, equivalently, that A is θ -closed.

SUFFICIENCY. Let $x \in X$. Assume that $\{x\}$ is not closed. Then $B = X \setminus \{x\}$ is not open and, trivially, B is θ -generalized closed since the only open superset of B is X itself. By (2), B is θ -closed and thus $\{x\}$ is θ -open. Since a singleton is θ -open if and only if it is closed, $\{x\}$ is closed. $\square$

The notion of a Λ -set and a generalized Λ -set in a topological space was introduced in [16]. By definition, a subset A of a topological space (X, τ) is called a Λ -set [16] if $A = A^\Lambda$, where $A^\Lambda = \cap \{U : U \supset A, U \in \tau\}$. Recall that A is called a generalized Λ -set [16] if $A^\Lambda \subseteq F$, whenever $A \subseteq F$ and F is τ -closed.

DEFINITION 5. (i) For a subset A of (X, τ) , we define A_θ^Λ as follows

$$A_\theta^\Lambda = \{x \in X : \text{Cl}_\theta\{x\} \cap A \neq \emptyset\}.$$

In [12], A_θ^Λ is denoted by $\ker_\theta A$.

(ii) A subset A of (X, τ) is called θ -generalized Λ -set (= θ -g- Λ -set) if $A_\theta^\Lambda \subseteq F$, whenever $A \subseteq F$ and F is closed in (X, τ) .

OBSERVATION 4.4. (i) Every G_δ -set is a Λ -set.

(ii) [12, Lem. 3.5(a)]. For any set $A \subseteq X$, $A \subseteq A^\Lambda \subseteq A_\theta^\Lambda \subseteq \text{Cl}_\theta(A)$.

(iii) Every θ -closed set is a Λ -set.

(iv) Every g -closed Λ -set is closed.

(v) Every θ -generalized Λ -set is a generalized Λ -set.

REMARK 4.5. (i) A Λ -set need not be θ -closed. Any singleton of an infinite space with the cofinite topology is a Λ -set (since the space is T_1) but none of the singletons is θ -closed.

(ii) A closed set need not be a Λ -set. In the Sierpinski space $(X = \{a, b\}, \tau = \{\emptyset, \{a\}, X\})$, the set $B = \{b\}$ is closed but B is not a Λ -set. However, in [16, Prop. 3.8], it was shown that in a topological space (X, τ) , every subset of X is a generalized Λ -set if and only if every closed set is a Λ -set.

(iii) A generalized Λ -set need not be θ -generalized Λ -set. In an infinite cofinite space X , as mentioned in Remark 4.5, every singleton is a Λ -set and, hence, a generalized Λ -set but none of the singletons is a θ -generalized Λ -set since the θ -closure of every singleton is X .

In [16], it was proved that in T_1 -spaces, every set is a Λ -set. Note that the converse is also true.

PROPOSITION 4.6. (i) A topological space (X, τ) is a T_1 -space if and only if every subset of X is a Λ -set.

(ii) A topological space (X, τ) is an R_0 -space if and only if every singleton of X is a generalized Λ -set.

PROOF. (i) Obvious.

(ii) In [9], Dube showed that a space is R_0 if and only if, for each closed set A , $A = A^\Lambda$. Thus, if X is R_0 , then for each singleton $\{x\}$ and each closed set F containing x , we have $\{x\} \subseteq \{x\}^\Lambda \subseteq F^\Lambda = F$. So, $\{x\}$ is a generalized Λ -set. For the reverse assume that $F \subseteq X$ is closed. For each $x \in F$, by assumption, $\{x\}^\Lambda \subseteq F$. Thus, $F^\Lambda = \cup_{x \in F} \{x\}^\Lambda \subseteq F$ according to [16, condition (2.5)]. This shows that $F = F^\Lambda$. $\square$

OBSERVATION 4.7. (i) A subset A of an R_1 -space X is generalized Λ -set if and only if A is θ -generalized Λ -set.

(ii) In Hausdorff spaces, every subset is a θ -generalized Λ -set.

(iii) A topological space X is Hausdorff if and only if X is a kc -space and every closed set of X is a θ -generalized Λ -set.

5. θ -g-continuous and θ -g-irresolute functions

DEFINITION 6. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

- (1) θ - g -continuous if $f^{-1}(V)$ is θ - g -closed in (X, τ) for every closed set V of (Y, σ) ,
- (2) θ - g -irresolute if $f^{-1}(V)$ is θ - g -closed in (X, τ) for every θ - g -closed set V of (Y, σ) .

OBSERVATION 5.1. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is strongly θ -continuous, then f is θ - g -continuous.

EXAMPLE 5.2. Let (X, τ) be the space in Example 3.2. Let $\sigma = \{\emptyset, \{b\}, X\}$. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function. Clearly, in the notion of Example 3.2, f is θ - g -continuous but f is not strongly θ -continuous, not even semi-continuous.

OBSERVATION 5.3. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be θ - g -continuous. Then f is g -continuous but not conversely.

EXAMPLE 5.4. Let (X, τ) be the space in Example 3.4. Let $\sigma = \{\emptyset, \{a, b\}, X\}$. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function. Clearly, f is continuous and hence g -continuous but as shown in Example 3.4, $A = \{c\} \notin \text{TGC}(X, \tau)$ and hence f is not θ - g -continuous.

Example 5.2 and Example 5.4 also show that continuity and θ - g -continuity are independent concepts. Thus, we have the following implications and none of them is reversible.

EXAMPLE 5.5. Let f be the function in Example 5.2. Let $\nu = \{\emptyset, \{c\}, X\}$. Let $g : (X, \sigma) \rightarrow (X, \nu)$ be the identity function. It is easily observed that g is also θ -generalized continuous. But the composition function $g \circ f : (X, \tau) \rightarrow (X, \nu)$ is not θ -generalized continuous since $\{a, b\} \notin \text{TGC}(X, \tau)$.

THEOREM 5.6. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is bijective, open and θ -generalized continuous, then f is θ - g -irresolute.

PROOF. Let $V \in \text{TGC}(Y)$ and let $f^{-1}(V) \subseteq O$, where $O \in \tau$. Clearly, $V \subseteq f(O)$. Since $f(O) \in \sigma$ and since $V \in \text{TGC}(Y)$, $\text{Cl}_\theta(V) \subseteq f(O)$ and thus $f^{-1}(\text{Cl}_\theta(V)) \subset O$. Since f is θ -generalized continuous and since $\text{Cl}_\theta(V)$ is closed in Y , $\text{Cl}_\theta(f^{-1}(\text{Cl}_\theta(V))) \subseteq O$ and hence $\text{Cl}_\theta(f^{-1}(V)) \subseteq O$. Therefore, $f^{-1}(V) \in \text{TGC}(X)$. Hence, f is θ - g -irresolute. $\square$

DEFINITION 7. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called θ -generalized closed if, for every closed set F of (X, τ) , $f(F)$ is θ - g -closed in (Y, σ) .

THEOREM 5.7. (i) Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be continuous and θ -generalized closed. Then, for a θ - g -closed set A of X , $f(A)$ is θ - g -closed in Y .

(ii) Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be strongly θ -continuous and closed. Then, f is θ - g -irresolute.

PROOF. (i) Left to the reader.

(ii) Let B be a θ - g -closed set of (Y, σ) and let $U \in \tau$ such that $f^{-1}(B) \subseteq U$. Put $H = \text{Cl}_\theta(f^{-1}(B)) \cap (X \setminus U)$. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is strongly θ -continuous if and only if $f : (X, \tau) \rightarrow (Y, \sigma)$ is (γ, id) -continuous in the sense of Ogata [22, Def. 4.12], where $\gamma : \tau \rightarrow \mathcal{P}(X)$ is the closure operation and $\text{id} : \sigma \rightarrow \mathcal{P}(Y)$ is the identity operation. Using [22, Prop. 4.13(i)] and the fact that $\text{Cl}_\gamma(E) = \text{Cl}_\theta(E)$ and $\text{Cl}_{\text{id}}(E) = \text{Cl}(E)$ for the closure operation γ , the identity operation id and the subset E , we get $f(H) \subseteq f(\text{Cl}_\theta(f^{-1}(B))) \cap f(X \setminus U) \subseteq \text{Cl}(f(f^{-1}(B))) \cap (X \setminus B) \subseteq \text{Cl}(B) \setminus B \subset \text{Cl}_\theta(B) \setminus B$. By Lemma 4.2, $f(H) = \emptyset$ since $f(H)$ is closed. We have $H = \emptyset$ and hence $\text{Cl}_\theta(f^{-1}(B)) \subseteq U$. Therefore, $f^{-1}(B) \in \text{TGC}(X, \tau)$. $\square$

COROLLARY 5.8. (i) Under the same assumptions of Theorem 5.6, if (X, τ) is $T_{1/2}$, then (Y, σ) is $T_{1/2}$.

(ii) Under the same assumptions of Theorem 5.7(ii), if (X, τ) is $T_{1/2}$ and $f : (X, \tau) \rightarrow (Y, \sigma)$ is surjective, then (Y, σ) is $T_{1/2}$.

PROPOSITION 5.9. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a θ -generalized continuous function and let H be a θ -closed subset of X . Then the restriction $f|_H : (H, \tau|_H) \rightarrow (Y, \sigma)$ is θ -generalized continuous.

PROOF. Let F be a closed subset of (Y, σ) . By Proposition 3.11, $H_1 = f^{-1}(F) \cap H$ is θ -generalized closed in (X, τ) . Then, by Theorem 3.13(ii), H_1 is θ - g -closed in $(H, \tau|_H)$. Since $(f|_H)^{-1}(F) = H_1$, $f|_H$ is θ - g -continuous. $\square$

Next, we offer the following “Pasting Lemma” for θ - g -continuous functions.

PROPOSITION 5.10. Let (X, τ) be a topological space such that $X = A \cup B$, where both $A, B \in \text{TGC}(X)$ and $A, B \in \tau$. Let $f : (A, \tau|_A) \rightarrow (Y, \sigma)$ and $g : (B, \tau|_B) \rightarrow (Y, \sigma)$ be θ -generalized continuous functions such that $f(x) = g(x)$ for every $x \in A \cap B$. Then the combination $\alpha : (X, \tau) \rightarrow (Y, \sigma)$ is θ -generalized continuous, where $\alpha(x) = f(x)$ for any $x \in A$ and $\alpha(y) = g(y)$ for any $y \in B$.

DEFINITION 8. A subset A of (X, τ) is called θ -generalized open (= θ - g -open) if its complement $X \setminus A$ is θ -generalized closed in (X, τ) .

THEOREM 5.11. (i) A subset A of (X, τ) is θ - g -open if and only if $F \subseteq \text{Int}_\theta(A)$, whenever $F \subset A$ and F is closed in (X, τ) .

(ii) If A is θ - g -open in (X, τ) and B is θ - g -open in (Y, σ) , then $A \times B$ is θ - g -open in the product space $(X \times Y, \tau \times \sigma)$.

PROOF. (i) Obvious.

(ii) Let F be a closed subset of $(X \times Y, \tau \times \sigma)$ such that $F \subseteq A \times B$. For each $(x, y) \in F$, $\text{Cl}(\{x\}) \times \text{Cl}(\{y\}) \subseteq \text{Cl}(F) = F \subseteq A \times B$. Then the two closed sets $\text{Cl}(\{x\})$ and $\text{Cl}(\{y\})$ are contained in A and B , respectively. By assumption, $\text{Cl}(\{x\}) \subseteq \text{Int}_\theta(A)$ and $\text{Cl}(\{y\}) \subseteq \text{Int}_\theta(B)$ hold. This implies that, for each $(x, y) \in F$, $(x, y) \in \text{Int}_\theta(A) \times \text{Int}_\theta(B) \subseteq \text{Int}_\theta(A \times B)$ and hence $F \subset \text{Int}_\theta(A \times B)$. By (i) it is clear that $A \times B$ is θ - g -open. $\square$

PROPOSITION 5.12. *The projection $p : (X \times Y, \tau \times \sigma) \rightarrow (X, \tau)$ is a θ -g-irresolute map.*

PROOF. By definition and Theorem 5.11(ii), for a θ -generalized closed set F of (X, τ) , $p^{-1}(x \setminus F) = (X \setminus F) \times Y$ is θ -g-open in $(X \times Y, \tau \times \sigma)$. Therefore, $P^{-1}(F) = F \times Y = X \times Y \setminus (p^{-1}(X \setminus F))$ is θ -generalized closed. $\square$

6. TGO-connected spaces. In 1991, Balachandran et al. [3] introduced a stronger form of connectedness called GO-connectedness. A set is called *g-open* [15] if its complement is g-closed.

DEFINITION 9. (cf. [15]). A topological space X is called *TGO-connected* (respectively, *GO-connected* [15]) if X cannot be written as a disjoint union of two nonempty θ -g-open (respectively, g-open) sets. A subset of X is called TGO-connected if it is connected as a subspace.

Clearly, every TGO-connected space is connected. The space in [3, Ex. 11] shows that there are connected spaces which are not TGO-connected. Since every θ -generalized closed set is g-closed, every GO-connected space is TGO-connected. Thus, we have the following implications and none of them is reversible.

$$\text{GO-connected} \Rightarrow \text{TGO-connected} \Rightarrow \text{Connected}$$

EXAMPLE 6.1. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$. Since $\{c\}$ is both g-closed and g-open, X is not GO-connected. Note that $\text{TGC}(X) = \{\emptyset, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$. Hence, X is TGO-connected.

OBSERVATION 6.2. (i) [3, Prop. 10]. *For a topological space (X, τ) , the following conditions are equivalent.*

- (1) X is TGO-connected;
- (2) the only subsets of X , which are both θ -g-open and θ -g-closed, are $\emptyset$ and X ;
- (3) each θ -generalized continuous function of X into a discrete space Y , with at least two points, is constant.

(ii) [3, Prop. 12]. *If (X, τ) is a $T_{1/2}$ -space, then the following conditions are equivalent*

- (1) X is GO-connected;
- (2) X is TGO-connected;
- (3) X is connected.

(iii) *A regular space X is GO-connected if and only if X is TGO-connected.*

(iv) *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a surjection. Then*

- (a) *If f is θ -generalized continuous and X is TGO-connected, then Y is connected.*
- (b) *If f is θ -g-irresolute and X is TGO-connected, then Y is TGO-connected.*

COROLLARY 6.3. *If the product space $(X \times Y, \tau \times \sigma)$ is TGO-connected, then its factor space (X, τ) is TGO-connected.*

THEOREM 6.4. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be θ -g-continuous. Then the image of every θ -closed, TGO-connected subset of (X, τ) is connected in (Y, σ) .*

PROOF. Let H be a θ -closed and TGO-connected set in (X, τ) . Then, by Proposition 5.9, the restriction of f to H , $f|_H : (H, \tau|_H) \rightarrow (Y, \sigma)$, is θ -g-continuous. For f , a function $r_H(f) : (H, \tau|_H) \rightarrow (f(H), \sigma|_{f(H)})$ is well defined by $(r_H(f))(x) = f(x)$ for any $x \in H$. Since $f|_H = j \circ r_H(f)$, where $j : (f(H), \tau|_{f(H)}) \rightarrow (Y, \sigma)$ is an inclusion. Then it is clear that $r_H(f)$ is θ -g-continuous. In fact, for an open set V of $(f(H), \sigma|_{f(H)})$, take an open set $G \in \tau$ such that $G \cap f(H) = V$. Then $r_H(f)^{-1}(V) = (f|_H)^{-1}(G)$ is θ -g-open. Now, by Observation 6.2(iv), $(f(H), \sigma|_{f(H)})$ is connected and hence $f(H)$ is a connected subset of (Y, σ) . $\square$

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