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SOME REMARKS ON THE ALGEBRAIC STRUCTURE OF THE FINITE COXETER GROUP F_4

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Abstract. We consider in this paper the algebraic structure and some properties of the finite Coxeter group F_4.

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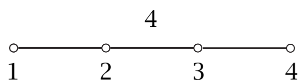
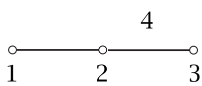
1. Introduction. The group F_4 is one of the irreducible Coxeter groups [9] defined by the presentation

$$F_4 = \Big\langle x_1, x_2, x_3, x_4 \mid x_i^2 = e, \quad 1 \le i \le 4 \\ (x_1x_2)^3 = (x_3x_4)^3 = (x_2x_3)^4 = (x_1x_3)^2 = (x_1x_4)^2 = (x_2x_4)^2 = e \Big\rangle. \tag{1}$$

It has the graph



It is obvious that the group B_3 whose graph is



is a subgroup of F_4. The order of B_3 is known to be 48 [4]. It is easy to see that the index of B_3 in F_4 is 24 and hence the order of F_4 is 1152.

2. The structure of F_4. We define F_4 by the presentation given in Section 1. We consider the symmetric group of degree 3 with the presentation

$$S_3 = \langle x, y \mid x^2 = y^2 = (xy)^3 = e \rangle. \tag{2}$$

We define the map $\theta : F_4 \longrightarrow S_3$, where

$$\theta(x_1) = x, \quad \theta(x_2) = y, \quad \theta(x_3) = \theta(x_4) = e. \tag{3}$$

It is easy to see that θ is an epimorphism and so $F_4/\ker\theta \cong S_3$. We use the Reidemeister-Schreier process to find a partition for $\ker\theta$.

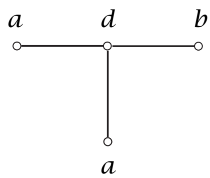
A Schreier transversal for $\ker\theta$ in F_4 is

$$U = \{e, x_1, x_2, x_1x_2, x_2x_1, x_1x_2x_1\}. \tag{4}$$

The process gives us the following partition for $\ker\theta$:

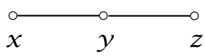
$$\begin{aligned}\ker\theta = \langle a,b,c,d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 \\ = (ad)^3 = (bd)^3 = (cd)^3 = (ac)^2 = e\rangle.\end{aligned} \tag{5}$$

Therefore, $\ker\theta$ is the Coxeter group D_4 whose graph is



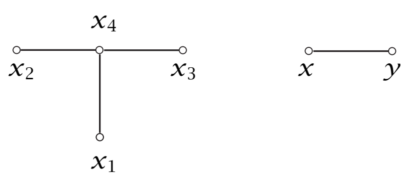
This shows that the group F_4 is the split extension of the Coxeter group D_4 by S_3.

Remark 1. To identify the structure of D_4, we consider the map $\theta : D_4 \longrightarrow S_4$, where D_4 is defined by the graph above and S_4 is defined by the graph



and $\theta(a) = x$, $\theta(d) = y$, $\theta(b) = z$, and $\theta(c) = y$. Using the Reidemeister-Schreier process, we find that $\ker\theta \cong Z_2^3$. Thus, D_4 is the split extension of Z_2^3 by S_4. An alternative method is given in [3], where D_n is shown to be the semi-direct product of Z_2^{n-1} by S_n.

Remark 2. A third method to show that $F \cong D_4 \rtimes S_3$ follows. We consider D_4 and S_3 as having the following graphs:



where $x = (12)$ and $y = (23)$. We consider the natural action of S_3 or D_4 defined as

$$(x_1,x_2,x_3,x_4)^x = (x_2,x_1,x_3,x_4) \quad \text{and} \quad (x_1,x_2,x_3,x_4)^y = (x_1,x_3,x_2,x_4). \tag{6}$$

We let E to be the split extension of D_4 by S_3 with this action. A presentation for E is

$$E = \langle x_1,x_2,x_3,x_4,x,y \mid \text{Relations of } D_4,\ \text{Relations of } S_3,\ \text{Action of } S_3 \text{ on } D_4\rangle. \tag{7}$$

(See [2].) Simple Tietze transformations show that $E \cong F_4$. Hence, $F_4 \cong D_4 \rtimes S_3$.

3. The derived series of F_4. We use the Reidemeister-Schreier process several times to find the derived series of F_4. Firstly, let F_4 have the presentation in Section 1. $F_4/F_4' \cong Z_2 \times Z_2$ and we find that $F_4' = \langle x, y \mid x^3 = y^3 = (x^{-1}y^{-1}xy)^2 = e \rangle$. The group $F_4'/F_4'' \cong Z_3 \times Z_3$ and we get $F_4'' = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (ac)^2 = (cd)^2 = (bd)^2 = (bdca)^2 = e \rangle$. Finally, $F_4''/F_4''' \cong Z_2^4$ and we find $F_4''' = Z_2$. Thus, we have proved that F_4 is solvable of derived length 4.

4. The center and the growth series of F_4. We have seen in Section 2 that $F_4 \cong D_4 \rtimes S_3$ and that $D_4 \cong Z_2^3 \rtimes S_4$. It is easy to see that the center of D_4 is Z_2 (in general, $Z(D_n) = Z_2$ if n is even and $\{e\}$ if n is odd [3]). Since $Z(S_3) = \{e\}$, we see that $Z(F_4) \subseteq Z(D_4) = Z_2$. Let $Z(D_4)$ be generated by g. From the Reidemeister-Schreier process, we can find g in terms of the generators of F_4 and show that it does not commute with any of them. Hence, $Z(F_4) = \{e\}$.

The growth series (in the sense of Gromov and Milnor) of F_4 is [5]

$$\gamma(F_4) = (1+t)^4(1+t^2)^2(1+t^4)(1-t+t^2)^2(1+t+t^2)^2(1-t^2+t^4). \tag{8}$$

The order of F_4 is obtained here as $\gamma(F_4)(1) = 2^4 \times 2^2 \times 2 \times 3^2 = 1152$.

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